

THE DOMAIN OF PARABOLICITY FOR THE MUSKAT PROBLEM

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ABSTRACT. We address the well-posedness of the Muskat problem in a periodic geometry and in a setting which allows us to consider general initial and boundary data, gravity effects, as well as surface tension effects. In the absence of surface tension we prove that the Rayleigh-Taylor condition identifies a domain of parabolicity for the Muskat problem. This property is used to establish the well-posedness of the problem. In the presence of surface tension effects the Muskat problem is of parabolic type for general initial and boundary data. As a bi-product of our analysis we obtain that Dirichlet-Neumann type operators associated with certain diffraction problems are negative generators of strongly continuous and analytic semigroups in the scale of small Hölder spaces.

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1. INTRODUCTION

We study the evolution of two vertically superposed (or horizontally adjacent) immiscible layers of Newtonian fluids with (possibly) different densities and viscosities in a two-dimensional periodic porous medium or Hele-Shaw cell when allowing for both gravity and surface tension effects. The pressure on the fixed flat boundary of the lower layer is prescribed and the upper fluid layer is assumed to be bounded from above by air at uniform pressure. This leads to a moving boundary problem for the interface between the two layers, the interface between the upper layer and the air, and the velocity potentials in the two fluid layers. The associated mathematical model is the Muskat problem, which was originally proposed in [31] as a model for the encroachment of water into an oil sand. It is given in (2.3) below in the absence of surface tension effects and accordingly in (7.2) when allowing for surface tension effects.

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Muskat problems have been studied extensively in the last two decades. Surface tension effects are considered in the papers [18, 20, 24, 28] where questions related to well-posedness (for small initial data) and stability properties of trivial, that is, circular or flat, and finger-shaped equilibria are addressed (see [16] for a classification of equilibria). There is a much larger list of references dealing with the Muskat problem without surface tension effects, the methods used in the studies being numerous and quite different. The well-posedness property is established in [35, 36] by using Newton's iteration method; the references [3, 11] use energy estimates (see also [6, 12–14] for the case of fluids with equal viscosities); [18, 20] rely on abstract parabolic theory and the continuous maximal regularity due to Da Prato and Grisvard [15]; [33] employs – in the absence of gravity effects – methods from complex analysis and a version of the Cauchy-Kowalewski theorem. Existence of solutions for nonregular initial data is shown in [5] by means of a fixed point argument. There are various interesting phenomena established for fluids with equal viscosities: global existence of strong and weak solutions for initial data which are bounded by explicit constants [10, 27], existence of initial data for which solutions turn over [7–9], or the absence of squirt or splash singularities [13, 23, 25].

An important role in the study of the Muskat problem is played by the Rayleigh-Taylor condition, which is a sign restriction on the jump of the gradient of the pressure in normal direction along an interface that separates two phases (see (2.7) below for more details) and was originally found within the linear theory [32]. In the absence of surface tension effects, the paper [20] was (one of) the first in which it was proved that the Muskat problem has, at least for small initial and boundary data, a parabolic character provided the Rayleigh-Taylor condition holds. For general initial data the well-posedness of the problem in different geometries is also implied by the Rayleigh-Taylor condition [3, 11, 33, 35, 36]. However, it is worth mentioning that the true character of the problem was not revealed in any of the just cited papers.

In this paper we now prove for arbitrary (sufficiently smooth) initial data that the Muskat problem with and without surface tension effects has a parabolic character. More precisely, when neglecting surface tension we establish the parabolicity of the problem provided the Rayleigh-Taylor condition holds. This enables us to use Da Prato and Grisvard's abstract parabolic theory, in particular continuous maximal regularity, in order to prove the well-posedness of this problem, cf. Theorem 2.1. Having two moving interfaces, we actually need to impose the Rayleigh-Taylor at each of them. We also show that the Muskat problem with surface tension is parabolic for arbitrary (sufficiently smooth) initial and boundary data, the corresponding well-posedness result being stated in Theorem 7.1. As a bi-product of our analysis we show in Proposition 5.8 and Remark 7.5 that Dirichlet-Neumann type operators associated with certain diffraction problems are negative generators of strongly continuous and analytic semigroups in the scale of small Hölder spaces.

It is worth to emphasize that the abstract parabolic setting mentioned above appears to be one of the few where the Muskat problem can be handled both with or without surface tension effects and with general boundary data. In addition, our analysis allows us to handle two fluids with possibly different viscosities or densities or even to neglect the effects of gravity (the latter being reasonable, for example, when the Hele-Shaw cell is not vertical, but horizontal or also in microfluidic models). In many studies these aspects were not taken into account.

The paper is organized as follows. In Section 2 we present the Muskat problem without surface tension and the first main result Theorem 2.1, whose proof requires some preparation. In Section 3 we first discuss the solvability of a general diffraction problem and recast the Muskat problem as a fully nonlinear and nonlocal evolution equation. We then show in Sections 4–6 that the Fréchet derivative of the operator associated with this evolution problem is an analytic generator. For this

we use localization techniques in the spirit of [21], but such that we keep the setting of periodic functions. The proof of Theorem 2.1 is then a consequence of this generator result. The well-posedness of the Muskat problem with surface tension effects is addressed in Section 7.

2. THE MUSKAT PROBLEM WITHOUT SURFACE TENSION EFFECTS

To set the stage we need some notation. In what follows \mathbb{S} denotes the unit circle $\mathbb{R}/(2\pi\mathbb{Z})$ meaning that functions depending on $x \in \mathbb{S}$ are 2π -periodic with respect to the real variable x . Given $m \in \mathbb{N}$ and $\beta \in (0, 1)$, the small Hölder space $h^{m+\beta}(\mathbb{S})$ stands for the closure of the smooth functions $C^\infty(\mathbb{S})$ in $C^{m+\beta}(\mathbb{S})$. It is well-known that $h^{m+\beta}(\mathbb{S})$ is a true subspace of the classical Hölder space $C^{m+\beta}(\mathbb{S})$, cf. e.g. [18], and that $C^r(\mathbb{S})$ is densely embedded in $h^s(\mathbb{S})$ if $r > s > 0$. Recall that $h^m(\mathbb{S}) = C^m(\mathbb{S})$ for $m \in \mathbb{N}$. Similarly, given two functions $\phi, \psi \in C(\mathbb{S})$ with $\phi(x) < \psi(x)$ for all $x \in \mathbb{S}$ and setting

$$\Omega := \Omega(\phi, \psi) := \{(x, y) : x \in \mathbb{S} \text{ and } \phi(x) < y < \psi(x)\}, \quad (2.1)$$

we denote by $h^{m+\beta}(\Omega)$ the closure of the smooth functions $C^\infty(\overline{\Omega})$ in $C^{m+\beta}(\overline{\Omega})$. As before, $C^r(\overline{\Omega})$ is densely embedded in $h^s(\Omega)$ if $r > s$ provided that $\phi, \psi \in h^s(\mathbb{S})$.

Let $\alpha \in (0, 1)$ and $d < 0$ be fixed constants and set

$$\mathcal{V} := \{(f, h) \in (h^{2+\alpha}(\mathbb{S}))^2 : d < f < h\}.$$

For each pair $(f, h) \in \mathcal{V}$ we define $\Omega(f) := \Omega(d, f)$ and $\Omega(f, h)$ according to (2.1). We look for a pair of functions $(f, h) : [0, T_0) \rightarrow \mathcal{V}$ with $T_0 > 0$ describing the evolution of the interfaces $\Gamma(f) := [y = f]$ and $\Gamma_h := [y = h]$ that bound two incompressible and immiscible Newtonian fluid layers in a porous medium and at constant temperature. At each time instant $t \in [0, T_0)$, the domain $\Omega(f(t))$ is assumed to be occupied by a fluid with density ρ_- and viscosity μ_- , respectively $\Omega(f(t), h(t))$ is the domain occupied by a second fluid with density ρ_+ and viscosity μ_+ . Note that neither the densities nor the viscosities need to be equal in what follows. We define the velocity potentials

$$u_\pm := p_\pm + g\rho_\pm y, \quad (2.2)$$

with g being the Earth's gravity and p_\pm the fluids' pressures with densities ρ_\pm . Our results hold also true when neglecting gravity, that is, when $g = 0$, so we assume $g \geq 0$ in the following. The velocity fields \vec{v}_\pm then obey Darcy's law, see [31],

$$\vec{v}_- = -\frac{k}{\mu_-} \nabla u_- \quad \text{in } \Omega(f) \quad \text{and} \quad \vec{v}_+ = -\frac{k}{\mu_+} \nabla u_+ \quad \text{in } \Omega(f, h),$$

and the incompressibility condition reads

$$\operatorname{div} v_+ = 0 \quad \text{in } \Omega(f) \quad \text{and} \quad \operatorname{div} v_- = 0 \quad \text{in } \Omega(f, h).$$

Here, $k > 0$ is a constant which stands for the permeability of the porous medium. Assuming the pressure on the boundary component $\Gamma_d := [y = d]$ to be known, in the absence of surface tension

effects the Muskat problem is the system of partial differential equations

$$\left\{ \begin{array}{ll} \Delta u_+ = 0 & \text{in } \Omega(f, h), \\ \Delta u_- = 0 & \text{in } \Omega(f), \\ \partial_t h = -k\mu_+^{-1}\sqrt{1+h'^2}\partial_\nu u_+ & \text{on } \Gamma(h), \\ u_+ = g\rho_+ h & \text{on } \Gamma(h), \\ u_- = b & \text{on } \Gamma_d, \\ u_+ - u_- = g(\rho_+ - \rho_-)f & \text{on } \Gamma(f), \\ \partial_t f = -k\mu_\pm^{-1}\sqrt{1+f'^2}\partial_\nu u_\pm & \text{on } \Gamma(f), \end{array} \right. \quad (2.3a)$$

governing the evolution of the fluids supplemented with the initial conditions

$$f(0) = f_0, \quad h(0) = h_0. \quad (2.3b)$$

We have additionally taken the pressure of the air to be constant zero, we assumed that the interfaces between the fluids move along with the fluids, and that the pressure is continuous along the interfaces. Given $\phi \in C^1(\mathbb{S})$, we have chosen $\nu := (-\phi', 1)/\sqrt{1+\phi'^2}$ to be the unit outward normal vector at the curve $[y = \phi]$.

The function $b = b(t, x)$ corresponds to the given pressure at the interface $[y = d]$ and is assumed to belong to the class

$$b \in C([0, T], h^{2+\alpha}(\mathbb{S})), \quad (2.4)$$

for some $T \in (0, \infty]$. Our main goal is to study the existence and uniqueness of *classical Hölder solutions* to the Muskat problem (2.3), that is, of tuples (f, h, u_+, u_-) with

$$\begin{aligned} (f, h) &\in C([0, T_0], \mathcal{V}) \cap C^1([0, T_0], (h^{1+\alpha}(\mathbb{S}))^2), \\ u_+(t) &\in h^{2+\alpha}(\Omega(f(t), h(t))), \quad u_-(t) \in h^{2+\alpha}(\Omega(f(t))) \end{aligned} \quad (2.5)$$

for all $t \in [0, T_0]$ with $T_0 \in (0, T]$, and which satisfy the equations of (2.3) pointwise.

Given $(f_0, h_0) \in \mathcal{V}$ and $b_0 := b(0) \in h^{2+\alpha}(\mathbb{S})$, Theorem 3.2 and Remark 3.1 below ensure that the diffraction problem

$$\left\{ \begin{array}{ll} \Delta u_+^0 = 0 & \text{in } \Omega(f_0, h_0), \\ \Delta u_-^0 = 0 & \text{in } \Omega(f_0), \\ u_+^0 = g\rho_+ h_0 & \text{on } \Gamma(h_0), \\ u_-^0 = b_0 & \text{on } \Gamma_d, \\ u_+^0 - u_-^0 = g(\rho_+ - \rho_-)f_0 & \text{on } \Gamma(f_0), \\ \mu_- \partial_\nu u_+^0 = \mu_+ \partial_\nu u_-^0 & \text{on } \Gamma(f_0), \end{array} \right. \quad (2.6)$$

possesses a unique solution $(u_+^0, u_-^0) \in h^{2+\alpha}(\Omega(f_0, h_0)) \times h^{2+\alpha}(\Omega(f_0))$. Letting p_+^0 and p_-^0 be the initial pressures determined, respectively, by u_+^0 and u_-^0 according to (2.2), we shall show that the conditions

$$\begin{aligned} \partial_\nu p_-^0 - \partial_\nu p_+^0 &< 0 \quad \text{on } \Gamma(f_0), \\ \partial_\nu p_+^0 &< 0 \quad \text{on } \Gamma(h_0) \end{aligned} \quad (2.7)$$

define a regime (for $(f_0, h_0, b_0) \in \mathcal{V} \times h^{2+\alpha}(\mathbb{S})$) where the Muskat problem (2.3) is parabolic. Since the air pressure is constant, the condition (2.7) expresses the Rayleigh-Taylor condition imposed at each interface as mentioned in the Introduction. To be more precise, we shall prove in Section 3 that the Muskat problem can be recast as a fully nonlinear abstract evolution equation for the interfaces f and h only, that is,

$$\partial_t(f, h) = \Phi(t, (f, h)),$$

which is of parabolic type when (2.7) holds. By parabolicity we mean that the Fréchet derivative $\partial_{(f,h)}\Phi(0, (f_0, h_0))$ is the generator of a strongly continuous and analytic semigroup. This property is the corner stone in our analysis and, together with the abstract parabolic theory due to Da Prato and Grisvard [15, 30], it enables us to establish the following well-posedness result for the Muskat problem without surface tension effects.

Theorem 2.1. *Let $g \geq 0$, $(f_0, h_0) \in \mathcal{V}$, and b be given such that (2.4) holds. Assume that the Rayleigh-Taylor conditions (2.7) are satisfied.*

Then, there exist a maximal existence time $T_0 := T_0(f_0, h_0) \in (0, T]$ and a unique classical Hölder solution (f, h, u_+, u_-) to (2.3) on $[0, T_0)$. Additionally, the solutions depend continuously on the initial data.

Remarks 2.2. (a) *Our analysis discloses that the Muskat problem is backwards parabolic when the Rayleigh-Taylor condition holds with reversed inequalities.*

(b) *When the fluids have the same viscosities, the last two equations of (2.6) show that the Rayleigh-Taylor condition on $\Gamma(f_0)$, i.e., the first condition in (2.7), is equivalent to $\rho_- > \rho_+$ (this is the case in [6, 12–14]). Hence, in this case the Muskat problem (2.3) is well-posed provided that*

$$\rho_- > \rho_+ \quad \text{and} \quad \partial_\nu p_+^0 < 0 \quad \text{on } \Gamma(h_0).$$

(c) *In the particular case $d = -1$, $b(0) \equiv c \in \mathbb{R}$, and $(f_0, h_0) \equiv (0, 1)$, the conditions (2.7) coincide with those found in [18, eq. (2.2)-(2.3)], that is,*

$$g\rho_+ > -\frac{c\mu_+}{\mu_-} \quad \text{and} \quad \frac{\mu_+ - \mu_-}{\mu_+ + \mu_-}(c - g\rho_+) + g(\rho_+ - \rho_-) < 0.$$

This shows in particular that the set of data $(f_0, h_0, b(0))$ for which the Rayleigh-Taylor conditions (2.7) are satisfied is not empty.

(d) *If gravity is neglected, that is, if $g = 0$, the Rayleigh-Taylor conditions (2.7) are equivalent to*

$$\begin{aligned} (\mu_+ - \mu_-)\partial_\nu p_+^0 &< 0 \quad \text{on } \Gamma(f_0), \\ \partial_\nu p_+^0 &< 0 \quad \text{on } \Gamma(h_0). \end{aligned} \tag{2.8}$$

As for Remarks 2.2 (d) we point out that if b_0 is zero or a negative function, then (2.8) cannot be satisfied. Indeed, if b_0 is the zero function, then both p_+^0 and p_-^0 are identically zero. If b_0 is a negative function, then p_-^0 is also negative since otherwise there exists $x_0 \in \mathbb{S}$ such that $p_-^0(x_0, f(x_0)) = \max_{\overline{\Omega}_-} p_-^0 \geq 0$, and so $\partial_\nu p_-^0 > 0$ at $(x_0, f(x_0))$ by Hopf's lemma. But p_+^0 is harmonic as well and not constant, therefore $p_+^0(x_0, f(x_0)) = \max_{\overline{\Omega}_+} p_+^0 \geq 0$, so that $\partial_\nu p_+^0 < 0$ at $(x_0, f(x_0))$ in contradiction to the last equation of (2.6). Hence, p_-^0 is negative implying $p_+^0 < 0$ on $\Gamma(f_0)$. By Hopf's lemma we find $\partial_\nu p_+^0 > 0$ on $\Gamma(h_0)$, and (2.8) is again not satisfied.

Lastly, if b_0 is a positive function, the previous arguments show that (2.8) is equivalent to

$$(\mu_+ - \mu_-)\partial_\nu p_+^0 < 0 \quad \text{on } \Gamma(f_0).$$

As p_+^0 attains its positive maximum on $\Gamma(f_0)$, we have that $\partial_\nu p_+^0$ is negative at least at one point on this interface implying that

$$\mu_+ < \mu_-. \tag{2.9}$$

It is easy to see that if b_0 is positive and constant and if also f_0 and h_0 are constant functions, then (2.9) is equivalent to (2.7). Thus, if b_0 is positive and constant we find, as in [33], that the Muskat

problem is well-posed for small initial data – that is, initial data close to constants in $(h^{2+\alpha}(\mathbb{S}))^2$ – when the more viscous fluid expands into the less viscous one.

The proof of Theorem 2.1 is postponed to the end of Section 6 as it requires several preparatory results that will be given in the subsequent sections.

3. THE EVOLUTION EQUATION

In order to solve problem (2.3) we re-write it as an abstract evolution equation on the unit circle. To do so we first transform system (2.3a) into a system of equations on fixed domains by using the unknown functions (f, h) . Let $\Omega_- := \mathbb{S} \times (-1, 0)$, $\Omega_+ := \mathbb{S} \times (0, 1)$, and define for each $(f, h) \in \mathcal{V}$ the mappings $\phi_f : \Omega_- \rightarrow \Omega(f)$ and $\phi_{(f,h)} : \Omega_+ \rightarrow \Omega(f, h)$ by setting

$$\phi_f(x, y) := (x, -dy + (1 + y)f(x)) \quad \text{and} \quad \phi_{(f,h)}(x, y) := (x, yh(x) + (1 - y)f(x)),$$

respectively. One easily checks that ϕ_f and $\phi_{(f,h)}$ are diffeomorphisms for all $(f, h) \in \mathcal{V}$. Each pair $(f, h) \in \mathcal{V}$ induces linear uniformly elliptic operators

$$\begin{aligned} \mathcal{A}(f) : h^{2+\alpha}(\Omega_-) &\rightarrow h^\alpha(\Omega_-), & v_- &\mapsto \Delta(v_- \circ \phi_f^{-1}) \circ \phi_f, \\ \mathcal{A}(f, h) : h^{2+\alpha}(\Omega_+) &\rightarrow h^\alpha(\Omega_+), & v_+ &\mapsto \Delta(v_+ \circ \phi_{(f,h)}^{-1}) \circ \phi_{(f,h)}, \end{aligned}$$

which depend, as bounded operators, real-analytically on f and h (see the formulae in the Appendix). Denote by tr_0 the trace operator with respect to $\Gamma_0 := \mathbb{S} \times \{0\}$. We associate with problem (2.3a) trace operators on Γ_0 ,

$$\begin{aligned} \mathcal{B}(f)v_- &:= k\mu_-^{-1} \text{tr}_0(\langle \nabla(v_- \circ \phi_f^{-1}) | (-f', 1) \rangle \circ \phi_f), & v_- &\in h^{2+\alpha}(\Omega_-), \\ \mathcal{B}(f, h)v_+ &:= k\mu_+^{-1} \text{tr}_0(\langle \nabla(v_+ \circ \phi_{(f,h)}^{-1}) | (-f', 1) \rangle \circ \phi_{(f,h)}), & v_+ &\in h^{2+\alpha}(\Omega_+), \end{aligned}$$

which, seen as bounded operators into $h^{1+\alpha}(\mathbb{S})$, depend real-analytically on f and h as well. Lastly, we define a boundary operator on Γ_1 , where $\Gamma_{\pm 1} := \mathbb{S} \times \{\pm 1\}$. Given $(f, h) \in \mathcal{V}$, we set

$$\mathcal{B}_1(f, h)v_+ := k\mu_+^{-1} \text{tr}_1(\langle \nabla(v_+ \circ \phi_{(f,h)}^{-1}) | (-h', 1) \rangle \circ \phi_{(f,h)}), \quad v_+ \in h^{2+\alpha}(\Omega_+),$$

where $\text{tr}_{\pm 1}$ is the trace operator with respect to $\Gamma_{\pm 1}$.

Remark 3.1. *Given $(f, h) \in \mathcal{V}$, the mappings*

$$\begin{aligned} [u_- \mapsto u_- \circ \phi_f] : h^{2+\alpha}(\Omega(f)) &\rightarrow h^{2+\alpha}(\Omega_-), \\ [u_+ \mapsto u_+ \circ \phi_{(f,h)}] : h^{2+\alpha}(\Omega(f, h)) &\rightarrow h^{2+\alpha}(\Omega_+), \end{aligned}$$

are isomorphisms.

Proof. See, for instance, the proof of [19, Lemma 1.2]. □

In view of Remark 3.1, it follows that (f, h, u_+, u_-) is a solution to (2.3) if and only if (f, h, v_+, v_-) with $v_+ := u_+ \circ \phi_{(f,h)}$ and $v_- := u_- \circ \phi_f$ is a classical Hölder solution to

$$\left\{ \begin{array}{ll} \mathcal{A}(f, h)v_+ = 0 & \text{in } \Omega_+, \\ \mathcal{A}(f)v_- = 0 & \text{in } \Omega_-, \\ \mathcal{B}(f, h)v_+ - \mathcal{B}(f)v_- = 0 & \text{on } \Gamma_0, \\ v_+ - v_- = g(\rho_+ - \rho_-)f & \text{on } \Gamma_0, \\ v_+ = g\rho_+h & \text{on } \Gamma_1, \\ v_- = b & \text{on } \Gamma_{-1}, \end{array} \right. \quad (3.1a)$$

with

$$\left\{ \begin{array}{ll} \partial_t h = -\mathcal{B}_1(f, h)v_+ & \text{on } \Gamma_1, \\ \partial_t f = -\mathcal{B}(f)v_- & \text{on } \Gamma_0, \end{array} \right. \quad (3.1b)$$

and

$$f(0) = f_0, \quad h(0) = h_0. \quad (3.1c)$$

The notion of classical Hölder solution to (3.1) is defined analogously to that for problem (2.3).

A diffraction problem in Hölder spaces. The system (3.1a) is an elliptic diffraction (or transmission) problem, problems of this type being highly relevant in many physical situations such as the study of multiphase dynamics. However, citable references on this topic are sparse. The main goal in this part is to establish the following result on the existence, uniqueness, and real-analytic dependence of solutions to (3.1a) on given $(f, h) \in \mathcal{V}$ and $b \in h^{2+\alpha}(\mathbb{S})$.

Theorem 3.2. *Given $(f, h) \in \mathcal{V}$ and $b \in h^{2+\alpha}(\mathbb{S})$, there exists a unique solution*

$$(v_+, v_-) := (v_+(f, h, b), v_-(f, h, b)) \in h^{2+\alpha}(\Omega_+) \times h^{2+\alpha}(\Omega_-)$$

to the diffraction problem (3.1a). Moreover, it holds that

$$[(f, h, b) \mapsto (v_+(f, h, b), v_-(f, h, b))] \in C^\omega(\mathcal{V} \times h^{2+\alpha}(\mathbb{S}), h^{2+\alpha}(\Omega_+) \times h^{2+\alpha}(\Omega_-)).$$

In the Hölder setting considered herein, problem (3.1a) can be accessed by using the celebrated Agmon-Douglis-Nirenberg estimates on solutions to elliptic boundary value systems presented in [1]. To prove Theorem 3.2 we first consider a particular boundary value problem for a linear elliptic system with coupled boundary conditions for which we establish the existence and uniqueness of solutions in the natural Hölder spaces. This is the context of the next proposition.

In the following $\partial_1 := \partial_x$, $\partial_2 := \partial_y$ and we identify both boundaries of Ω_+ with the unit circle \mathbb{S} .

Proposition 3.3. *Let $\mathcal{L}_k := a_{ij}^{(k)}\partial_{ij} + b_i^{(k)}\partial_i + c^{(k)}$ be uniformly elliptic operators with coefficients $a_{ij}^{(k)}$, $b_i^{(k)}$, $c^{(k)} \in C^\alpha(\overline{\Omega}_+)$ for $i, j, k = 1, 2$ and such that $c^{(k)} \leq 0$. Additionally, let $\mathcal{B}_k := \beta_i^{(k)}\text{tr}_0 \partial_i + \gamma^{(k)}$ be two boundary operators such that $\beta_i^{(k)}, \gamma^{(k)} \in C^{1+\alpha}(\mathbb{S})$, $\beta_2^{(k)} > 0$, and $\gamma^{(k)} \leq 0$, $k = 1, 2$. Given $F_1, F_2 \in C^\alpha(\overline{\Omega}_+)$, $\varphi_1 \in C^{1+\alpha}(\mathbb{S})$, and $\varphi_2, \varphi_3, \varphi_4 \in C^{2+\alpha}(\mathbb{S})$, the boundary value problem*

$$\left\{ \begin{array}{ll} \mathcal{L}_1 w_1 = F_1 & \text{in } \Omega_+, \\ \mathcal{L}_2 w_2 = F_2 & \text{in } \Omega_+, \end{array} \right. \quad (3.2a)$$

and

$$\left\{ \begin{array}{ll} \mathcal{B}_1 w_1 + \mathcal{B}_2 w_2 = \varphi_1 & \text{on } \Gamma_0, \\ w_1 - w_2 = \varphi_2 & \text{on } \Gamma_0, \end{array} \right. \quad \left\{ \begin{array}{ll} w_1 = \varphi_3 & \text{on } \Gamma_1, \\ w_2 = \varphi_4 & \text{on } \Gamma_1, \end{array} \right. \quad (3.2b)$$

possesses a unique solution $(w_1, w_2) \in (C^{2+\alpha}(\overline{\Omega}_+))^2$.

Proof. We discuss uniqueness first. To this end, let (w_1, w_2) be a solution to (3.2) with right-hand sides replaced all by zero. If $\max_{\overline{\Omega}_+} w_1 > 0$ (the case $\min_{\overline{\Omega}_+} w_1 < 0$ is similar), the weak elliptic maximum principle ensures that

$$\max_{\overline{\Omega}_+} w_1 = \max_{\overline{\Omega}_+} w_2 = w_1(x, 0) = w_2(x, 0)$$

for some $x \in \mathbb{S}$. Applying Hopf's lemma at $(x, 0)$ for both w_1 and w_2 yields $\partial_y w_1(x, 0) < 0$ and $\partial_y w_2(x, 0) < 0$. This is in contradiction to the equation $\mathcal{B}_1 w_1 + \mathcal{B}_2 w_2 = 0$ on Γ_0 . Hence, (w_1, w_2) has to be the zero solution, and therefore (3.2) has at most one solution $(w_1, w_2) \in (C^{2+\alpha}(\overline{\Omega}_+))^2$.

For the existence part, we consider a family of operators $\{\mathcal{T}_\tau := (\mathcal{T}_\tau^1, \dots, \mathcal{T}_\tau^6)\}_{\tau \in [0, 1]} \subset \mathcal{L}(\mathbb{X}, \mathbb{Y})$, with $\mathbb{X} := (C^{2+\alpha}(\overline{\Omega}_+))^2$, $\mathbb{Y} := (C^\alpha(\overline{\Omega}_+))^2 \times C^{1+\alpha}(\mathbb{S}) \times (C^{2+\alpha}(\mathbb{S}))^3$, and

$$\mathcal{T}_\tau(w_1, w_2) := \begin{pmatrix} (1 - \tau)\mathcal{L}_1 w_1 + \tau\Delta w_1 \\ (1 - \tau)\mathcal{L}_2 w_2 + \tau\Delta w_2 \\ (1 - \tau)(\mathcal{B}_1 w_1 + \mathcal{B}_2 w_2) + \tau \operatorname{tr}_0 \partial_y (w_1 + w_2) \\ \operatorname{tr}_0 (w_1 - w_2) \\ \operatorname{tr}_1 w_1 \\ \operatorname{tr}_1 w_2 \end{pmatrix}$$

for $\tau \in [0, 1]$ and $(w_1, w_2) \in (C^{2+\alpha}(\overline{\Omega}_+))^2$. We observe that $[\tau \mapsto \mathcal{T}_\tau] \in C([0, 1], \mathcal{L}(\mathbb{X}, \mathbb{Y}))$. Moreover, by considering two suitable Dirichlet problems the invertibility of \mathcal{T}_1 can be reduced to the solvability of the equation $\mathcal{T}_1(z_1, z_2) = (0, 0, \varphi, 0, 0, 0, 0)$ for arbitrary $\varphi \in C^{1+\alpha}(\mathbb{S})$. Indeed, letting (z_1, z_2) denote the solution to the equation $\mathcal{T}_1(z_1, z_2) = (0, 0, \varphi, 0, 0, 0, 0)$ for $\varphi := \varphi_1 - \operatorname{tr}_0 \partial_y (\tilde{w}_1 + \tilde{w}_2)$, and setting

$$\tilde{w}_1 := (\Delta, \operatorname{tr}_0, \operatorname{tr}_1)^{-1}(F_1, \varphi_2, \varphi_3) \quad \text{and} \quad \tilde{w}_2 := (\Delta, \operatorname{tr}_0, \operatorname{tr}_1)^{-1}(F_2, 0, \varphi_4),$$

it is easy to see that $(w_1, w_2) := (z_1 + \tilde{w}_1, z_2 + \tilde{w}_2) \in \mathbb{X}$ satisfies $\mathcal{T}_1(w_1, w_2) = (F_1, F_2, \varphi_1, \varphi_2, \varphi_3, \varphi_4)$. The solution to $\mathcal{T}_1(z_1, z_2) = (0, 0, \varphi, 0, 0, 0, 0)$ is $z_1 = z_2 := (\Delta, \operatorname{tr}_0 \partial_y, \operatorname{tr}_1)^{-1}(0, \varphi/2, 0) \in C^{2+\alpha}(\overline{\Omega}_+)$. Hence, we have shown that \mathcal{T}_1 is invertible. If we find a constant $C > 0$, such that

$$\|(w_1, w_2)\|_{\mathbb{X}} \leq C \|\mathcal{T}_\tau(w_1, w_2)\|_{\mathbb{Y}} \quad \text{for all } \tau \in [0, 1] \text{ and } (w_1, w_2) \in \mathbb{X}, \quad (3.3)$$

then by the method of continuity, cf. e.g. [26], we conclude that \mathcal{T}_0 is an isomorphism, which is the claim of the proposition.

We are left to establish (3.3). To this end, we show that \mathcal{T}_τ corresponds to a uniformly elliptic system that satisfies the Complementing Condition in the sense of [1] on both boundary components Γ_0 and Γ_1 . Because the equations in Ω_+ are decoupled, it is easy to see that $(\mathcal{T}_\tau^1, \mathcal{T}_\tau^2)$ defines a uniformly elliptic system for each $\tau \in [0, 1]$. Additionally, the boundary conditions defined by $(\mathcal{T}_\tau^5, \mathcal{T}_\tau^6)$ are of Dirichlet type, and therefore the Complementing Condition on Γ_1 is straightforward. To verify the Complementing Condition on Γ_0 we modify the operators \mathcal{T}_τ^i , $1 \leq i \leq 4$ as follows: we identify the principle parts $\mathcal{T}_\tau^{\pi, i}$, $1 \leq i \leq 4$, freeze their coefficients at an arbitrary $P \in \Gamma_0$, and replace (∂_1, ∂_2) by $(\xi, -i\partial_t)$ with $0 \neq \xi \in \mathbb{R}$. Doing this, we arrive at the initial value problem

$$\left\{ \begin{array}{ll} v_1'' - iA_1^{(1)} v_1' - A_2^{(1)} v_1 = 0, & \text{for } t > 0, \\ v_2'' - iA_1^{(2)} v_2' - A_2^{(2)} v_2 = 0, & \text{for } t > 0, \\ v_1(0) = v_2(0), & \\ i[(1 - \tau)\beta_2^{(1)}(P) + \tau]v_1'(0) + [(1 - \tau)\beta_2^{(2)}(P) + \tau]v_2'(0) \\ - (1 - \tau)\xi(\beta_1^{(1)}(P)v_1(0) + \beta_1^{(2)}(P)v_2(0)) = 0, & \end{array} \right. \quad (3.4)$$

where

$$A_1^{(k)} := -\frac{2(1-\tau)a_{12}^{(k)}(P)\xi}{(1-\tau)a_{22}^{(k)}(P)+\tau}, \quad A_2^{(k)} := \frac{((1-\tau)a_{11}^{(k)}(P)+\tau)\xi^2}{(1-\tau)a_{22}^{(k)}(P)+\tau}, \quad k=1,2.$$

The Complementing Condition is satisfied if and only if the only bounded solution (v_1, v_2) of (3.4) is the zero solution. It is readily seen that

$$v_k(t) = \gamma_1^{(k)} e^{i\delta_1^{(k)}t} e^{-\delta_2^{(k)}t} + \gamma_2^{(k)} e^{i\delta_1^{(k)}t} e^{\delta_2^{(k)}t}, \quad k=1,2,$$

with $\delta_1^{(k)} := A_1^{(k)}/2$ and $\delta_2^{(k)} := \sqrt{A_2^{(k)} - (A_1^{(k)})^2/4} > 0$. The boundedness of v_1, v_2 entails that $\gamma_2^{(1)} = \gamma_2^{(2)} = 0$. Moreover, the equation $v_1(0) = v_2(0)$ implies that $\gamma_1^{(1)} = \gamma_1^{(2)}$. Finally, assuming $\gamma_1^{(1)} \neq 0$, we find from the last equation of (3.4) that necessarily

$$\delta_2^{(1)}((1-\tau)\beta_2^{(1)}(P)+\tau) + \delta_2^{(2)}((1-\tau)\beta_2^{(2)}(P)+\tau) = 0.$$

However, this last equation cannot hold true as $\delta_2^{(k)}$ and $\beta_2^{(k)}(P), k=1,2$, are positive constants. Hence, the Complementing Condition is also satisfied on Γ_0 .

We may use now Theorem 9.3 and argue similarly as in the subsequent Remark 2 in [1] to conclude, together with the uniqueness result established at the beginning of the proof, that there exists a constant $C > 0$ such that the estimate (3.3) holds. This completes the proof. \square

Using Proposition 3.3, we obtain the unique solvability of certain diffraction problems within the natural Hölder spaces.

Corollary 3.4. *Let $\mathcal{L}_\pm := a_{ij}^\pm \partial_{ij} + b_i^\pm \partial_i + c^\pm$ be uniformly elliptic operators with coefficients $a_{ij}^\pm, b_i^\pm, c^\pm \in C^\alpha(\overline{\Omega}_\pm)$ and such that $c^\pm \leq 0$ for $i, j = 1, 2$. Moreover, let $\mathcal{B}_\pm := \beta_i^\pm \text{tr}_0 \partial_i + \gamma^\pm$ be boundary operators such that $\beta_i^\pm, \gamma^\pm \in C^{1+\alpha}(\mathbb{S})$, $\beta_2^\pm > 0$, and $\gamma^\pm \leq 0$. Then, given $F_\pm \in C^\alpha(\overline{\Omega}_\pm)$, $\varphi_1 \in C^{1+\alpha}(\mathbb{S})$, and $\varphi_2, \varphi_3, \varphi_4 \in C^{2+\alpha}(\mathbb{S})$, the diffraction problem*

$$\left\{ \begin{array}{ll} \mathcal{L}_+ v_+ = F_+ & \text{in } \Omega_+, \\ \mathcal{L}_- v_- = F_- & \text{in } \Omega_-, \\ \mathcal{B}_+ v_+ - \mathcal{B}_- v_- = \varphi_1 & \text{on } \Gamma_0, \\ v_+ - v_- = \varphi_2 & \text{on } \Gamma_0, \\ v_+ = \varphi_3 & \text{on } \Gamma_1, \\ v_- = \varphi_4 & \text{on } \Gamma_{-1}, \end{array} \right. \quad (3.5)$$

possesses a unique solution $(v_+, v_-) \in C^{2+\alpha}(\overline{\Omega}_+) \times C^{2+\alpha}(\overline{\Omega}_-)$. In particular, there exists a constant $C > 0$ such that

$$\|v_+\|_{2+\alpha} + \|v_-\|_{2+\alpha} \leq C \left(\|F_+\|_\alpha + \|F_-\|_\alpha + \|\varphi_1\|_{1+\alpha} + \sum_{i=2}^4 \|\varphi_i\|_{2+\alpha} \right). \quad (3.6)$$

Proof. The mapping $\phi : \Omega_+ \rightarrow \Omega_-$ defined by $\phi(y) = -y$, $y \in \Omega_+$ is a smooth diffeomorphism. Therefore, if (v_+, v_-) solves the system (3.5), then $w_1 := v_+$ and $w_2 = v_- \circ \phi$, is a solution to (3.2) with $\mathcal{L}_1 := \mathcal{L}_+$, $\mathcal{L}_2 := [w \mapsto (\mathcal{L}_-(w \circ \phi^{-1})) \circ \phi]$, $\mathcal{B}_1 := \mathcal{B}_+$, $\mathcal{B}_2 := [w \mapsto -\mathcal{B}_-(w \circ \phi^{-1})]$, $F_1 := F_+$, and $F_2 := F_- \circ \phi$. The desired claim follows now directly from Proposition 3.3. \square

We are now in the position to prove Theorem 3.2.

Proof of Theorem 3.2. Given $(f, h) \in \mathcal{V}$, it follows from Corollary 3.4 that the mapping

$$(v_+, v_-) \mapsto \begin{pmatrix} \mathcal{A}(f, h)v_+ \\ \mathcal{A}(f)v_- \\ \mathcal{B}(f, h)v_+ - \mathcal{B}(f)v_- \\ \text{tr}_0(v_+ - v_-) \\ \text{tr}_1 v_+, \\ \text{tr}_{-1} v_- \end{pmatrix} \quad (3.7)$$

defines an isomorphism between $C^{2+\alpha}(\overline{\Omega}_+) \times C^{2+\alpha}(\overline{\Omega}_-)$ and $C^\alpha(\overline{\Omega}_+) \times C^\alpha(\overline{\Omega}_-) \times C^{1+\alpha}(\mathbb{S}) \times (C^{2+\alpha}(\mathbb{S}))^3$. Since $(f, h) \in \mathcal{V}$, a density argument shows that the operator defined by (3.7) is a isomorphism also when acting between the corresponding small Hölder spaces $h^{2+\alpha}(\Omega_+) \times h^{2+\alpha}(\Omega_-)$ and $h^\alpha(\Omega_+) \times h^\alpha(\Omega_-) \times h^{1+\alpha}(\mathbb{S}) \times (h^{2+\alpha}(\mathbb{S}))^3$. Because the differential operators and the right-hand sides of the equations of (3.1a) depend in a real-analytic way on $(f, h, b) \in \mathcal{V} \times h^{2+\alpha}(\mathbb{S})$, the claim of Theorem 3.2 is now obvious. \square

The evolution equation. With the identification $\Gamma_i = \mathbb{S}$ for $i \in \{-1, 0, 1\}$ and by using Theorem 3.2, the problem (3.1a)-(3.1b) can now be reformulated as an abstract fully nonlinear and nonlocal evolution equation

$$\partial_t(f, h) = \Phi(t, (f, h)), \quad (3.8)$$

where $\Phi : [0, T] \times \mathcal{V} \rightarrow (h^{1+\alpha}(\mathbb{S}))^2$ is the operator $\Phi = (\Phi_1, \Phi_2)$ defined by

$$\begin{aligned} \Phi_1(t, (f, h)) &:= -\mathcal{B}(f)v_-(f, h, b(t)), \\ \Phi_2(t, (f, h)) &:= -\mathcal{B}_1(f, h)v_+(f, h, b(t)), \end{aligned} \quad (3.9)$$

with (v_+, v_-) denoting the solution operator introduced in Theorem 3.2. We note that

$$\Phi \in C([0, T] \times \mathcal{V}, (h^{1+\alpha}(\mathbb{S}))^2) \quad \text{and} \quad \Phi(t, \cdot) \in C^\omega(\mathcal{V}, (h^{1+\alpha}(\mathbb{S}))^2) \quad \text{for } t \in [0, T]. \quad (3.10)$$

Let $(f_0, h_0) \in \mathcal{V}$ and set

$$b_0 := b(0) \in h^{2+\alpha}(\mathbb{S}).$$

Our aim is to apply the existence result [30, Theorem 8.4.1] to (3.8) for which we need to show that

$$-\partial_{(f, h)}\Phi(0, (f_0, h_0)) \in \mathcal{H}((h^{2+\alpha}(\mathbb{S}))^2, (h^{1+\alpha}(\mathbb{S}))^2), \quad (3.11)$$

that is, $\partial_{(f, h)}\Phi(0, (f_0, h_0))$ seen as an unbounded operator in $(h^{1+\alpha}(\mathbb{S}))^2$ with domain of definition $(h^{2+\alpha}(\mathbb{S}))^2$ is the generator of a strongly continuous analytic semigroup in $\mathcal{L}((h^{1+\alpha}(\mathbb{S}))^2)$. This generator property will be established in the sections to follow for $(f_0, h_0) \in \mathcal{V}$ and $b_0 \in h^{2+\alpha}(\mathbb{S})$ for which (2.7) is satisfied.

Given $(f_*, h_*) \in \mathcal{V}$, the operator $\partial_{(f, h)}\Phi(0, (f_*, h_*)) \in \mathcal{L}((h^{2+\alpha}(\mathbb{S}))^2, (h^{1+\alpha}(\mathbb{S}))^2)$ can be written in matrix form

$$\partial_{(f, h)}\Phi(0, (f_*, h_*)) = \begin{pmatrix} \partial_f \Phi_1(0, (f_*, h_*)) & \partial_h \Phi_1(0, (f_*, h_*)) \\ \partial_f \Phi_2(0, (f_*, h_*)) & \partial_h \Phi_2(0, (f_*, h_*)) \end{pmatrix},$$

where, according to the definition (3.9), we have

$$\begin{aligned} \partial_f \Phi_1(0, (f_*, h_*))[f] &= -\partial_f \mathcal{B}(f_*)[f]v_-(f_*, h_*, b_0) - \mathcal{B}(f_*)\partial_f v_-(f_*, h_*, b_0)[f], \\ \partial_h \Phi_1(0, (f_*, h_*))[h] &= -\mathcal{B}(f_*)\partial_h v_-(f_*, h_*, b_0)[h], \\ \partial_f \Phi_2(0, (f_*, h_*))[h] &= -\partial_h \mathcal{B}_1(f_*, h_*)[h]v_+(f_*, h_*, b_0) - \mathcal{B}_1(f_*, h_*)\partial_h v_+(f_*, h_*, b_0)[h], \end{aligned} \quad (3.12)$$

for $(f, h) \in (h^{2+\alpha}(\mathbb{S}))^2$. Additionally, $(w_+[f], w_-[f]) := (\partial_f v_+(f_*, h_*, b_0)[f], \partial_f v_-(f_*, h_*, b_0)[f])$ is the solution to the diffraction problem

$$\left\{ \begin{array}{ll} \mathcal{A}(f_*, h_*)w_+[f] = -\partial_f \mathcal{A}(f_*, h_*)[f]v_+^* & \text{in } \Omega_+, \\ \mathcal{A}(f_*)w_-[f] = -\partial_f \mathcal{A}(f_*)[f]v_-^* & \text{in } \Omega_-, \\ \mathcal{B}(f_*, h_*)w_+[f] - \mathcal{B}(f_*)w_-[f] = -\partial_f \mathcal{B}(f_*, h_*)[f]v_+^* + \partial_f \mathcal{B}(f_*)[f]v_-^* & \text{on } \Gamma_0, \\ w_+[f] - w_-[f] = g(\rho_+ - \rho_-)f & \text{on } \Gamma_0, \\ w_+[f] = 0 & \text{on } \Gamma_1, \\ w_-[f] = 0 & \text{on } \Gamma_{-1}, \end{array} \right. \quad (3.13)$$

and $(W_+[h], W_-[h]) := (\partial_h v_+^*(f_*, h_*, b_0)[h], \partial_h v_-^*(f_*, h_*, b_0)[h])$ solves the diffraction problem

$$\left\{ \begin{array}{ll} \mathcal{A}(f_*, h_*)W_+[h] = -\partial_h \mathcal{A}(f_*, h_*)[h]v_+^* & \text{in } \Omega_+, \\ \mathcal{A}(f_*)W_-[h] = 0 & \text{in } \Omega_-, \\ \mathcal{B}(f_*, h_*)W_+[h] - \mathcal{B}(f_*)W_-[h] = -\partial_h \mathcal{B}(f_*, h_*)[h]v_+^* & \text{on } \Gamma_0, \\ W_+[h] - W_-[h] = 0 & \text{on } \Gamma_0, \\ W_+[h] = g\rho_+h & \text{on } \Gamma_1, \\ W_-[h] = 0 & \text{on } \Gamma_{-1}. \end{array} \right. \quad (3.14)$$

In (3.13) and (3.14) we set

$$(v_+^*, v_-^*) = (v_+, v_-)(f_*, h_*, b_0). \quad (3.15)$$

According to [2, Theorem I.1.6.1 and Remark I.1.6.2], (3.11) is satisfied provided the diagonal operators satisfy

$$-\partial_f \Phi_1(0, (f_0, h_0)) \in \mathcal{H}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})), \quad (3.16)$$

$$-\partial_h \Phi_2(0, (f_0, h_0)) \in \mathcal{H}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})), \quad (3.17)$$

and provided the following property holds for the off-diagonal operator: for each $\varepsilon > 0$ there exists $K_0 = K_0(\varepsilon) > 0$ such that

$$\|\partial_h \Phi_1(0, (f_0, h_0))[h]\|_{1+\alpha} \leq \varepsilon \|h\|_{2+\alpha} + K_0 \|h\|_{1+\alpha} \quad \text{for all } h \in h^{2+\alpha}(\mathbb{S}). \quad (3.18)$$

So, to establish (3.11) various computations are needed. In Section 4 we first prove (3.18) based on Schauder estimates for diffraction problems as presented in Corollary 3.4. The proofs of (3.16) and (3.17) respectively, are given in Sections 5 (see Theorem 5.7) and 6 (see Theorem 6.5) for which we use localization techniques in the spirit of [21] (see also [17, 22, 37]). However, our localization techniques are quite different from those therein as we do not consider problems in the halfplane, and our results are sharper, e.g. see Theorem 6.5 and [37, Theorem 14].

4. AN OFF-DIAGONAL OPERATOR

The main goal of this section is to establish the property (3.18). This is a consequence of the following lemma where (3.18) is established for general $(f_*, h_*) \in \mathcal{V}$.

Lemma 4.1. *Let $(f_*, h_*) \in \mathcal{V}$. Given $\varepsilon \in (0, 1)$, there exists $K_0 = K_0(\varepsilon) > 0$ such that*

$$\|\partial_h \Phi_1(0, (f_*, h_*))[h]\|_{1+\alpha} \leq \varepsilon \|h\|_{2+\alpha} + K_0 \|h\|_{1+\alpha} \quad \text{for all } h \in h^{2+\alpha}(\mathbb{S}). \quad (4.1)$$

Proof. The proof is based on estimates for boundary value problems for elliptic systems, cf. (3.6) and [1]. Let (f_n, h_n, b_n) be a sequence in $(C^\infty(\mathbb{S}))^3$ which converges towards (f_*, h_*, b_0) in $(h^{2+\alpha}(\mathbb{S}))^3$ and such that $(f_n, h_n) \in \mathcal{V}$ for all n . We then have

$$\begin{aligned} \|\partial_h \Phi_1(0, (f_*, h_*))[h]\|_{1+\alpha} &\leq C \|\text{tr}_0(\nabla \partial_h v_-(f_*, h_*, b_0)[h])\|_{1+\alpha} \\ &\leq C \|\text{tr}_0 \nabla (\partial_h v_-(f_n, h_n, b_n)[h] - \partial_h v_-(f_*, h_*, b_0)[h])\|_{1+\alpha} + C \|\text{tr}_0(\nabla \partial_h v_-(f_n, h_n, b_n)[h])\|_{1+\alpha} \\ &\leq C \|\partial_h v_-(f_n, h_n, b_n) - \partial_h v_-(f_*, h_*, b_0)\|_{\mathcal{L}(h^{2+\alpha}(\mathbb{S}), h^{2+\alpha}(\Omega_-))} \|h\|_{2+\alpha} \\ &\quad + C \|\text{tr}_0 \nabla (\partial_h v_-(f_n, h_n, b_n)[h])\|_{1+\alpha} =: I_1 + I_2 \end{aligned} \quad (4.2)$$

for all $h \in h^{2+\alpha}(\mathbb{S})$. Let $\varepsilon > 0$ be given. In view of Theorem 3.2, we can choose n large enough to guarantee that

$$I_1 \leq \frac{\varepsilon}{2} \|h\|_{2+\alpha}. \quad (4.3)$$

We are now left to estimate the term I_2 for a fixed n such that (4.3) holds. To this end, we let $(W_+^n[h], W_-^n[h]) := (\partial_h v_+(f_n, h_n, b_n)[h], \partial_h v_-(f_n, h_n, b_n)[h])$ denote the solution of (3.14) when replacing (f_*, h_*, b_0) by (f_n, h_n, b_n) and v_+^* by $v_+^n := v_+(f_n, h_n, b_n)$. Because (f_n, h_n, b_n) is smooth, it follows from [1, Theorem 9.3], by arguing as in the proof of Theorem 3.2, that v_+^n is a smooth function up to the boundary of Ω_+ . In particular, $v_+^n \in h^{3+\alpha}(\Omega_+)$. We now split the solution $(W_+^n[h], W_-^n[h])$ as

$$(W_+^n[h], W_-^n[h]) = (W_+^1, W_-^1) + (W_+^2, W_-^2), \quad (4.4)$$

where (W_+^1, W_-^1) is the solution to

$$\left\{ \begin{array}{ll} \mathcal{A}(f_n, h_n)W_+^1 = -\partial_h \mathcal{A}(f_n, h_n)[h]v_+^n & \text{in } \Omega_+, \\ \mathcal{A}(f_n)W_-^1 = 0 & \text{in } \Omega_-, \\ \mathcal{B}(f_n, h_n)W_+^1 - \mathcal{B}(f_n)W_-^1 = 0 & \text{on } \Gamma_0, \\ W_+^1 - W_-^1 = 0 & \text{on } \Gamma_0, \\ W_+^1 = 0 & \text{on } \Gamma_1, \\ W_-^1 = 0 & \text{on } \Gamma_{-1}, \end{array} \right. \quad (4.5)$$

respectively (W_+^2, W_-^2) solves

$$\left\{ \begin{array}{ll} \mathcal{A}(f_n, h_n)W_+^2 = 0 & \text{in } \Omega_+, \\ \mathcal{A}(f_n)W_-^2 = 0 & \text{in } \Omega_-, \\ \mathcal{B}(f_n, h_n)W_+^2 - \mathcal{B}(f_n)W_-^2 = -\partial_h \mathcal{B}(f_n, h_n)[h]v_+^n & \text{on } \Gamma_0, \\ W_+^2 - W_-^2 = 0 & \text{on } \Gamma_0, \\ W_+^2 = g\rho_+ h & \text{on } \Gamma_1, \\ W_-^2 = 0 & \text{on } \Gamma_{-1}. \end{array} \right. \quad (4.6)$$

In order to estimate W_-^2 we note that

$$\partial_h \mathcal{B}(f_n, h_n)[h] = -k\mu_+^{-1} \frac{(1 + f_n'^2)h}{(h_n - f_n)^2} \text{tr}_0 \partial_y,$$

and therefore, the right-hand side of the third equation of (4.6) belongs to $h^{2+\alpha}(\mathbb{S})$. In virtue of [1, Theorem 9.3], there exists a constant $C > 0$ (which depends on the previously fixed n) such that on the subdomain $(1/2)\Omega_- := \mathbb{S} \times (-1/2, 0)$ of Ω_- we have

$$\|W_-^2\|_{2+\alpha}^{(1/2)\Omega_-} \leq C(\|\partial_h \mathcal{B}(f_n, h_n)[h]v_+^n\|_{1+\alpha} + \|W_+^2\|_0 + \|W_-^2\|_0) \quad (4.7)$$

for all $h \in h^{2+\alpha}(\mathbb{S})$. Let $\alpha' \in (0, \alpha)$ be fixed. Recalling (3.6), we have

$$\|W_+^2\|_0 + \|W_-^2\|_0 \leq \|W_+^2\|_{2+\alpha'} + \|W_-^2\|_{2+\alpha'} \leq C(\|\partial_h \mathcal{B}(f_n, h_n)[h]v_+^n\|_{1+\alpha'} + \|h\|_{2+\alpha'}),$$

and together with (4.7) we end up with

$$\|W_-^2\|_{2+\alpha}^{(1/2)\Omega_-} \leq C(\|\partial_h \mathcal{B}(f_n, h_n)[h]v_+^n\|_{1+\alpha} + \|h\|_{2+\alpha'}) \leq C\|h\|_{2+\alpha'}. \quad (4.8)$$

Using the following interpolation property of the small Hölder spaces (e.g. see [30])

$$(h^r(\mathbb{S}), h^s(\mathbb{S}))_\theta = h^{(1-\theta)r + \theta s}(\mathbb{S}) \quad \text{for } \theta \in (0, 1) \text{ and } (1-\theta)r + \theta s \notin \mathbb{N}, \quad (4.9)$$

where $(\cdot, \cdot)_\theta = (\cdot, \cdot)_{\theta, \infty}^0$ is the real interpolation functor introduced by Da Prato and Grisvard [15], we infer from (4.8) and Young's inequality that there exists a constant $K(\varepsilon)$ such that

$$\|\mathrm{tr}_0 \nabla W_-^2\|_{1+\alpha} \leq \|W_-^2\|_{2+\alpha}^{(1/2)\Omega_-} \leq \frac{\varepsilon}{4} \|h\|_{2+\alpha} + K(\varepsilon) \|h\|_{1+\alpha} \quad (4.10)$$

for all $h \in h^{2+\alpha}(\mathbb{S})$.

Thanks to (4.10), we are left to prove a similar estimate for $\|\mathrm{tr}_0 \nabla W_-^1\|_{1+\alpha}$. To this end, we choose $\delta \in (0, 1)$ and a function $\chi := \chi_\delta \in C^\infty([0, 1])$ such that $0 \leq \chi \leq 1$, $\chi = 1$ on $(0, \delta/8)$, and $\chi = 0$ on $[\delta/4, 1]$. Note that (3.6) and the regularity $\mathcal{A} \in C^\omega(\mathcal{V}, \mathcal{L}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})))$ implies that, for a fixed $\alpha' \in (0, \alpha)$,

$$\|W_-^1\|_{2+\alpha'} + \|W_+^1\|_{2+\alpha'} \leq C\|h\|_{2+\alpha'} \quad (4.11)$$

for all $h \in h^{2+\alpha}(\mathbb{S})$. Using the more compact notation $\mathcal{A}(f_n, h_n) = a_{ij}\partial_{ij} + b_2\partial_2$, the pair $(W_+^1\chi, W_-^1)$ solves the system (4.5), but with the first equation replaced by the elliptic equation

$$\mathcal{A}(f_n, h_n)(W_+^1\chi) = -\chi\partial_h \mathcal{A}(f_n, h_n)[h]v_+^n + 2a_{12}\partial_1 W_+^1\chi' + a_{22}(\partial_2 W_+^1\chi' + W_+^1\chi'') + b_2 W_+^1\chi'$$

in Ω_+ . We now use (3.6) and (4.11) to get

$$\begin{aligned} \|W_-^1\|_{2+\alpha} &\leq C\left(\|\chi\partial_h \mathcal{A}(f_n, h_n)[h]v_+^n\|_\alpha + \|(2a_{12}\partial_1 W_+^1 + b_2 W_+^1 + a_{22}\partial_2 W_+^1)\chi' + a_{22}W_+^1\chi''\|_\alpha\right) \\ &\leq C\|\chi\partial_h \mathcal{A}(f_n, h_n)[h]v_+^n\|_\alpha + C(\delta)\|W_+^1\|_{1+\alpha} \\ &\leq C\|yh''\chi\|_\alpha + C(\delta)\|h\|_{2+\alpha'} \\ &\leq C\|y\chi\|_0\|h\|_{2+\alpha} + C(\delta)\|h\|_{2+\alpha'}. \end{aligned} \quad (4.12)$$

Choosing δ such that $C\|y\chi\|_0 < \varepsilon/4$, we infer from (4.12) that there exists a constant $K(\varepsilon)$ such that

$$\|\mathrm{tr}_0 \nabla W_-^1\|_{1+\alpha} \leq \|W_-^1\|_{2+\alpha}^{\Omega_-} \leq (\varepsilon/4)\|h\|_{2+\alpha} + K(\varepsilon)\|h\|_{1+\alpha} \quad (4.13)$$

for all $h \in h^{2+\alpha}(\mathbb{S})$. Gathering now (4.2), (4.3), (4.4), (4.10), and (4.13) we have established the desired estimate (4.1). \square

5. THE FIRST DIAGONAL OPERATOR

In this section we prove that $\partial_f \Phi_1(0, (f_0, h_0))$ is the generator of a strongly continuous and analytic semigroup, hence (3.16), when (f_0, h_0) and b_0 are such that the first inequality in (2.7) holds. This is stated in Theorem 5.7. We start in Lemma 5.1 by identifying the “leading order part” $\partial_f \Phi_1^\pi(f_*, h_*)$ of $\partial_f \Phi_1(0, (f_*, h_*))$ for a general $(f_*, h_*) \in \mathcal{V}$. This reduces our task (see (5.4) and [2, Theorem I.1.3.1 (ii)]) to showing the generator property merely for $\partial_f \Phi_1^\pi(f_0, h_0)$, and thus allows us to neglect several “lower order terms” in the quite involved computations to follow. Following this step, we locally approximate the principal part $\partial_f \Phi_1^\pi(0, (f_*, h_*))$ by Fourier multipliers for (f_*, h_*) sufficiently close to (f_0, h_0) and possessing additional regularity, cf. Theorem 5.5. These multipliers

are then shown to be generators of strongly continuous analytic semigroups, where the constants in the resolvent estimates are uniform with respect to certain variables, cf. Lemma 5.6. This uniformity property is essential when establishing the desired Theorem 5.7 by means of Lemma 5.2 and a continuity argument.

Lemma 5.1. *Given $(f_*, h_*) \in \mathcal{V}$, let $\partial_f \Phi_1^\pi(f_*, h_*) \in \mathcal{L}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}))$ denote the operator defined by*

$$\partial_f \Phi_1^\pi(f_*, h_*)[f] := -\frac{k}{\mu_-} \left(\frac{2f'_* \operatorname{tr}_0 \partial_y v_-^*}{f_* - d} - \operatorname{tr}_0 \partial_x v_-^* \right) f' - \mathcal{B}(f_*) w_-^\pi[f], \quad f \in h^{2+\alpha}(\mathbb{S}), \quad (5.1)$$

with (v_+^*, v_-^*) defined in (3.15) and where $(w_+^\pi[f], w_-^\pi[f])$ denotes the solution to the problem

$$\left\{ \begin{array}{ll} \mathcal{A}_0^\pi(f_*, h_*) w_+^\pi[f] = \frac{\operatorname{tr}_0 \partial_y v_+^*}{h_* - f_*} f'' & \text{in } \Omega_+, \\ \mathcal{A}_0^\pi(f_*) w_-^\pi[f] = \frac{\operatorname{tr}_0 \partial_y v_-^*}{f_* - d} f'' & \text{in } \Omega_-, \\ \mathcal{B}(f_*, h_*) w_+^\pi[f] - \mathcal{B}(f_*) w_-^\pi[f] = -\partial_f \mathcal{B}^\pi(f_*, h_*)[f] v_+^* + \partial_f \mathcal{B}^\pi(f_*)[f] v_-^* & \text{on } \Gamma_0, \\ w_+^\pi[f] - w_-^\pi[f] = g(\rho_+ - \rho_-) f & \text{on } \Gamma_0, \\ w_+^\pi[f] = 0 & \text{on } \Gamma_1, \\ w_-^\pi[f] = 0 & \text{on } \Gamma_{-1}, \end{array} \right. \quad (5.2)$$

with

$$\begin{aligned} \mathcal{A}_0^\pi(f_*) &:= \partial_{xx} - \frac{2f'_*}{f_* - d} \partial_{xy} + \frac{f_*'^2 + 1}{(f_* - d)^2} \partial_{yy}, & \mathcal{A}_0^\pi(f_*, h_*) &:= \partial_{xx} - \frac{2f'_*}{h_* - f_*} \partial_{xy} + \frac{f_*'^2 + 1}{(h_* - f_*)^2} \partial_{yy}, \\ \partial_f \mathcal{B}^\pi(f_*)[f] &:= \frac{k}{\mu_-} \left(\frac{2f'_*}{f_* - d} \operatorname{tr}_0 \partial_y - \operatorname{tr}_0 \partial_x \right) f', & \partial_f \mathcal{B}^\pi(f_*, h_*)[f] &:= \frac{k}{\mu_+} \left(\frac{2f'_*}{h_* - f_*} \operatorname{tr}_0 \partial_y - \operatorname{tr}_0 \partial_x \right) f'. \end{aligned} \quad (5.3)$$

Then, given $\varepsilon \in (0, 1)$, there exists $K_1 = K_1(\varepsilon) > 0$ such that

$$\|\partial_f \Phi_1(0, (f_*, h_*))[f] - \partial_f \Phi_1^\pi(f_*, h_*)[f]\|_{1+\alpha} \leq \varepsilon \|f\|_{2+\alpha} + K_1 \|f\|_{1+\alpha} \quad \text{for all } f \in h^{2+\alpha}(\mathbb{S}). \quad (5.4)$$

Moreover, $\partial_f \Phi_1^\pi \in C^\omega(\mathcal{V}, \mathcal{L}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})))$.

Proof. The regularity property follows by using similar arguments as in the proof of Theorem 3.2. In order to prove (5.4), let $\varepsilon \in (0, 1)$ be given and $\alpha' \in (0, \alpha)$ be fixed. We choose $\delta \in (0, 1)$ and a cut-off function $\chi := \chi_\delta \in C^\infty([-1, 1])$ such that $0 \leq \chi \leq 1$, $\chi = 1$ for $|y| \leq \delta/2$, and $\chi = 0$ for $|y| \geq \delta$. Using the same notation as in Section 3 (see (3.13)) we have from (3.12), (5.1), and the formula for $\partial_f \mathcal{B}(f_*)$ in the Appendix A

$$\begin{aligned} \|\partial_f \Phi_1(0, (f_*, h_*))[f] - \partial_f \Phi_1^\pi(f_*, h_*)[f]\|_{1+\alpha} &\leq C(\|f\|_{1+\alpha} + \|\mathcal{B}(f_*)(w_-[f] - w_-^\pi[f])\|_{1+\alpha}) \\ &\leq C(\|f\|_{1+\alpha} + \|\operatorname{tr}_0 \nabla(w_-[f] - w_-^\pi[f])\|_{1+\alpha}) \\ &\leq C(\|f\|_{1+\alpha} + \|\chi(w_-[f] - w_-^\pi[f])\|_{2+\alpha}^{\Omega_-}), \end{aligned} \quad (5.5)$$

where C is independent of δ . We now observe that

$$(u_+, u_-) := (w_+^\pi[f], w_-^\pi[f]) - (w_+[f], w_-[f])$$

solves a diffraction problem of the form

$$\left\{ \begin{array}{ll} \mathcal{A}_0^\pi(f_*, h_*)u_+ = a_0^+ f + a_1^+ f' + ya_2^+ ((1-y)\partial_y v_+^* - \text{tr}_0 \partial_y v_+^*) f'' \\ \quad + y(b_0^+ \partial_{xy} w_+[f] + b_1^+ \partial_{yy} w_+[f]) + b_2^+ \partial_y w_+[f] & \text{in } \Omega_+, \\ \mathcal{A}_0^\pi(f_*)u_- = a_0^- f + a_1^- f' + ya_2^- ((1+y)\partial_y v_-^* - \text{tr}_0 \partial_y v_-^*) f'' \\ \quad + y(b_0^- \partial_{xy} w_-[f] + b_1^- \partial_{yy} w_-[f]) + b_2^- \partial_y w_-[f] & \text{in } \Omega_-, \\ \mathcal{B}(f_*, h_*)u_+ - \mathcal{B}(f_*)u_- = cf & \text{on } \Gamma_0, \\ u_+ - u_- = 0 & \text{on } \Gamma_0, \\ u_+ = 0 & \text{on } \Gamma_1, \\ u_- = 0 & \text{on } \Gamma_{-1}, \end{array} \right.$$

with functions $a_i^\pm, b_i^\pm \in h^\alpha(\Omega_\pm), 0 \leq i \leq 2$, and $c \in h^{1+\alpha}(\mathbb{S})$. Therefore, $(u_+^\chi, u_-^\chi) := \chi(u_+, u_-)$ is the solution to

$$\left\{ \begin{array}{ll} \mathcal{A}_0^\pi(f_*, h_*)u_+^\chi = \chi \mathcal{A}_0^\pi(f_*, h_*)u_+ - \frac{2f_*'}{h_* - f_*} \chi' \partial_x u_+ + \frac{f_*'^2 + 1}{(h_* - f_*)^2} (2\chi' \partial_y u_+ + \chi'' u_+) & \text{in } \Omega_+, \\ \mathcal{A}_0^\pi(f_*)u_-^\chi = \chi \mathcal{A}_0^\pi(f_*)u_- - \frac{2f_*'}{f_* - d} \chi' \partial_x u_- + \frac{f_*'^2 + 1}{(f_* - d)^2} (2\chi' \partial_y u_- + \chi'' u_-) & \text{in } \Omega_-, \\ \mathcal{B}(f_*, h_*)u_+^\chi - \mathcal{B}(f_*)u_-^\chi = cf & \text{on } \Gamma_0, \\ u_+^\chi - u_-^\chi = 0 & \text{on } \Gamma_0, \\ u_+^\chi = 0 & \text{on } \Gamma_1, \\ u_-^\chi = 0 & \text{on } \Gamma_{-1}. \end{array} \right. \quad (5.6)$$

We can estimate the solutions to the systems (5.2) and (5.6) by using (3.6) and obtain that

$$\begin{aligned} \|u_-^\chi\|_{2+\alpha}^{\Omega_-} &\leq C \left(\|\chi \mathcal{A}_0^\pi(f_*, h_*)u_+\|_{\alpha}^{\Omega_+} + \|\chi \mathcal{A}_0^\pi(f_*)u_-\|_{\alpha}^{\Omega_-} + \|f\|_{1+\alpha} \right) + C(\delta) (\|u_+\|_{2+\alpha'} + \|u_-\|_{2+\alpha'}) \\ &\leq C \left(\|\chi((1-y)\partial_y v_+^* - \text{tr}_0 \partial_y v_+^*) f''\|_{\alpha}^{\Omega_+} + \|\chi((1+y)\partial_y v_-^* - \text{tr}_0 \partial_y v_-^*) f''\|_{\alpha}^{\Omega_-} \right. \\ &\quad \left. + \|\chi y(b_0^+ \partial_{xy} w_+[f] + b_1^+ \partial_{yy} w_+[f])\|_{\alpha}^{\Omega_+} + \|\chi y(b_0^- \partial_{xy} w_-[f] + b_1^- \partial_{yy} w_-[f])\|_{\alpha}^{\Omega_-} \right) \\ &\quad + C(\delta) \|f\|_{2+\alpha'} \\ &\leq C \left(\|\chi((1-y)\partial_y v_+^* - \text{tr}_0 \partial_y v_+^*)\|_0^{\Omega_+} + \|\chi((1+y)\partial_y v_-^* - \text{tr}_0 \partial_y v_-^*)\|_0^{\Omega_-} \right. \\ &\quad \left. + \|\chi y\|_0^{\Omega_+} + \|\chi y\|_0^{\Omega_-} \right) \|f\|_{2+\alpha} + C(\delta) \|f\|_{2+\alpha'}. \end{aligned}$$

Recalling the definition of $\chi = \chi_\delta$ and choosing $\delta > 0$ such that

$$\|\chi((1-y)\partial_y v_+^* - \text{tr}_0 \partial_y v_+^*)\|_0^{\Omega_+} + \|\chi((1+y)\partial_y v_-^* - \text{tr}_0 \partial_y v_-^*)\|_0^{\Omega_-} + \|\chi y\|_0^{\Omega_+} + \|\chi y\|_0^{\Omega_-} < \frac{\varepsilon}{2C},$$

the desired estimate (5.4) follows from the interpolation property (4.9). \square

Let (f_0, h_0) and b_0 be such that the first inequality of (2.7) holds. We are now left to show that the principal part $\partial_f \Phi_1^\pi(f_0, h_0)$ of $\partial_f \Phi_1(0, (f_0, h_0))$ is the generator of a strongly continuous and analytic semigroup in $\mathcal{L}(h^{1+\alpha}(\mathbb{S}))$. In view of the classical result [2, Theorem I.1.2.2], one has $-\partial_f \Phi_1^\pi(f_0, h_0) \in \mathcal{H}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}))$ if and only if there exist constants $\kappa_1 \geq 1$ and $\omega_1 > 0$ such that $-\partial_f \Phi_1^\pi(f_0, h_0) \in \mathcal{H}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}), \kappa_1, \omega_1)$, that is,

$$\omega_1 - \partial_f \Phi_1^\pi(f_0, h_0) \in \text{Isom}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}))$$

and

$$\kappa_1^{-1} \leq \frac{\|(\lambda - \partial_f \Phi_1^\pi(f_0, h_0))f\|_{1+\alpha}}{|\lambda| \cdot \|f\|_{1+\alpha} + \|f\|_{2+\alpha}} \leq \kappa_1 \quad \text{for all } \operatorname{Re} \lambda \geq \omega_1 \text{ and } 0 \neq f \in h^{2+\alpha}(\mathbb{S}).$$

To this end, for $\sigma > 0$, let S_σ denote the set consisting of those $(f_*, h_*) \in \mathcal{V}$ satisfying the inequalities

$$\begin{aligned} (1) \quad & \sigma < \min\{f_* - d, h_* - f_*\}, \quad \|f_*\|_2 + \|h_*\|_2 < \sigma^{-1}, \\ (2) \quad & g(\rho_- - \rho_+) > \sigma + \frac{\operatorname{tr}_0 \partial_y v_-^*}{h_* - f_*} - \frac{\operatorname{tr}_0 \partial_y v_+^*}{f_* - d}, \\ (3) \quad & \|\operatorname{tr}_0 \partial_y v_+^*\|_0 + \|\operatorname{tr}_0 \partial_y v_-^*\|_0 < \sigma^{-1}, \end{aligned}$$

where (v_+^*, v_-^*) is defined in (3.15). Since the functions (f_0, h_0) and b_0 are chosen such that the first inequality in (2.7) is satisfied, we may choose σ such that $(f_0, h_0) \in S_\sigma$.

Lemma 5.2. *Let $\sigma > 0$ be such that $(f_0, h_0) \in S_\sigma$. Assume that there exists a constant $\tilde{\kappa}_1 := \tilde{\kappa}_1(\sigma)$ and for each $(f_*, h_*) \in (h^{3+\alpha}(\mathbb{S}))^2 \cap S_\sigma \cap \mathbb{B}_{(h^{2+\alpha}(\mathbb{S}))^2}((f_0, h_0), \sigma)$ there exists a further constant $\tilde{\omega}_1 > 0$ with the property that*

$$-\partial_f \Phi_1^\pi(f_*, h_*) \in \mathcal{H}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}), \tilde{\kappa}_1, \tilde{\omega}_1).$$

Then

$$-\partial_f \Phi_1^\pi(f_0, h_0) \in \mathcal{H}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})).$$

Proof. Using the regularity assertions in Theorem 3.2 and Lemma 5.1 together with the density of $h^{3+\alpha}(\mathbb{S})$ in $h^{2+\alpha}(\mathbb{S})$, we find $(f_*, h_*) \in (h^{3+\alpha}(\mathbb{S}))^2 \cap S_\sigma \cap \mathbb{B}_{(h^{2+\alpha}(\mathbb{S}))^2}((f_0, h_0), \sigma)$ such that

$$\|\partial_f \Phi_1^\pi(f_*, h_*) - \partial_f \Phi_1^\pi(f_0, h_0)\|_{\mathcal{L}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}))} < 1/2\tilde{\kappa}_1.$$

The perturbation result [2, Theorem I.1.3.1 (i)] implies

$$-\partial_f \Phi_1^\pi(f_0, h_0) \in \mathcal{H}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}), 2\tilde{\kappa}_1, \tilde{\omega}_1),$$

which yields the desired claim. \square

Remark 5.3. *We will see in the proof of Theorem 5.7 that, in contrast to κ_1 , the constants ω_1 in Lemma 5.2 appear to depend on the $\|\cdot\|_{3+\alpha}$ -norm of (f_*, h_*) . Whence, for (f_*, h_*) close to (f_0, h_0) , these constants may become large.*

We are now left to establish the assumptions of Lemma 5.2 for some sufficiently small σ . Therefore, we pick an arbitrary $(f_*, h_*) \in (h^{3+\alpha}(\mathbb{S}))^2 \cap S_\sigma$ and introduce a parameter $\tau \in [0, 1]$ which will enable us to continuously transform the leading order part $\partial_f \Phi_1^{\pi*} := \partial_f \Phi_1^\pi(f_*, h_*)$ of $\partial_f \Phi_1(0, (f_*, h_*))$ into a (negative) Dirichlet-Neumann map. This will allow us to use the continuation method and prove, by relying on the properties of this Dirichlet-Neumann map, that large positive real numbers belong to the resolvent set of $\partial_f \Phi_1^\pi(f_*, h_*)$. We emphasize that this construction uses to a large extent the additional regularity of (f_*, h_*) , as the mappings (v_+^*, v_-^*) introduced in (3.15) possess additional regularity close to the boundary Γ_0 in this case. More precisely, for each $\tau \in [0, 1]$ we introduce the operator $\partial_f \Phi_{1,\tau}^{\pi*} \in \mathcal{L}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}))$ by

$$\partial_f \Phi_{1,\tau}^{\pi*}[f] := -\frac{\tau k}{\mu_-} \left(\frac{2f'_* \operatorname{tr}_0 \partial_y v_-^*}{f_* - d} - \operatorname{tr}_0 \partial_x v_-^* \right) f' - \mathcal{B}(f_*) w_{-\tau}^\pi[f], \quad f \in h^{2+\alpha}(\mathbb{S}), \quad (5.7)$$

with (v_+^*, v_-^*) being defined in (3.15) and $(w_{+\tau}^\pi[f], w_{-\tau}^\pi[f])$ denoting the solution to

$$\left\{ \begin{array}{ll} \mathcal{A}_0^\pi(f_*, h_*) w_{+\tau}^\pi[f] = \frac{\tau \operatorname{tr}_0 \partial_y v_+^*}{h_* - f_*} f'' & \text{in } \Omega_+, \\ \mathcal{A}_0^\pi(f_*) w_{-\tau}^\pi[f] = \frac{\tau \operatorname{tr}_0 \partial_y v_-^*}{f_* - d} f'' & \text{in } \Omega_-, \\ \mathcal{B}(f_*, h_*) w_{+\tau}^\pi[f] - \mathcal{B}(f_*) w_{-\tau}^\pi[f] = -\tau \partial_f \mathcal{B}^\pi(f_*, h_*)[f] v_+^* + \tau \partial_f \mathcal{B}^\pi(f_*)[f] v_-^* & \text{on } \Gamma_0, \\ w_{+\tau}^\pi[f] - w_{-\tau}^\pi[f] = -\left[g(\rho_- - \rho_+) + (1 - \tau) \left(\frac{\operatorname{tr}_0 \partial_y v_+^*}{h_* - f_*} - \frac{\operatorname{tr}_0 \partial_y v_-^*}{f_* - d} \right) \right] f & \text{on } \Gamma_0, \\ w_{+\tau}^\pi[f] = 0 & \text{on } \Gamma_1, \\ w_{-\tau}^\pi[f] = 0 & \text{on } \Gamma_{-1}. \end{array} \right. \quad (5.8)$$

As $(f_*, h_*) \in (h^{3+\alpha}(\mathbb{S}))^2 \cap S_\sigma$, a density argument together with [1, Theorem 9.3] implies that the transformed potentials (v_+^*, v_-^*) satisfy

$$\frac{\operatorname{tr}_0 \partial_y v_-^*}{f_* - d} - \frac{\operatorname{tr}_0 \partial_y v_+^*}{h_* - f_*} \in h^{2+\alpha}(\mathbb{S}), \quad (5.9)$$

and therefore $(w_{+\tau}^\pi[f], w_{-\tau}^\pi[f])$ is well-defined, cf. Corollary 3.4. For $\tau = 1$ we recover the leading order part $\partial_f \Phi_{1,1}^{\pi*} = \partial_f \Phi_1^\pi(f_*, h_*)$ of $\partial_f \Phi_1(0, (f_*, h_*))$, while $\partial_f \Phi_{1,0}^{\pi*}$ is the above mentioned Dirichlet-Neumann map. Our method uses localization techniques based on a suitable partition of unity allowing us to keep up the setting of periodic functions.

Partition of unity. For each $p \geq 3$ there exists a family of functions $\{\Pi_j^p\}_{1 \leq j \leq 2^{p+1}} \subset C^\infty(\mathbb{S}, [0, 1])$ such that

- (i) $\operatorname{supp} \Pi_j^p = \cup_{n \in \mathbb{Z}} (2\pi n + I_j^p)$, where $I_j^p := [j - 5/3, j - 1/3]\pi/2^p$;
- (ii) $\sum_{j=1}^{2^{p+1}} \Pi_j^p = 1$ in $C(\mathbb{S})$.

The center of the interval I_j^p for $1 \leq j \leq 2^{p+1}$ is denoted by $x_j^p := (j - 1)\pi/2^p$. We call the family $\{\Pi_j^p\}_{1 \leq j \leq 2^{p+1}}$ a p -partition of unity.

Moreover, let $\{\chi_j^p\}_{1 \leq j \leq 2^{p+1}} \subset C^\infty(\mathbb{S}, [0, 1])$ be a corresponding family of functions such that

- (a) $\operatorname{supp} \chi_j^p = \cup_{n \in \mathbb{Z}} (2\pi n + J_j^p)$ and $I_j^p \subset J_j^p$;
- (b) $\chi_j^p = 1$ on I_j^p .

We can achieve that $J_j^p = I_{j-1}^p \cup I_j^p \cup I_{j+1}^p$, where $I_0^p := I_{2^{p+1}}^p$ and $I_{2^{p+1}+1}^p := I_1^p$, so that I_j^p and J_j^p have the same center x_j^p .

The following remark is a simple exercise.

Remark 5.4. Given $k, p \in \mathbb{N}$ with $p \geq 3$ and $\alpha \in (0, 1)$, the mapping

$$f \mapsto \max_{1 \leq j \leq 2^{p+1}} \|\Pi_j^p f\|_{k+\alpha}$$

defines a norm on $h^{k+\alpha}(\mathbb{S})$ which is equivalent to the $\|\cdot\|_{k+\alpha}$ -norm.

The following perturbation type result stays at the core of our analysis.

Theorem 5.5. Let $\sigma > 0$ be such that $(f_0, h_0) \in S_\sigma$ and let $\mu > 0$ and $\alpha' \in (0, \alpha)$ be given. Then, given $(f_*, h_*) \in (h^{3+\alpha}(\mathbb{S}))^2 \cap S_\sigma$, there exist an integer $p \geq 3$, a p -partition of unity $\{\Pi_j^p\}_{1 \leq j \leq 2^{p+1}}$,

and a constant $K_2 = K_2(p)$, and for each $\tau \in [0, 1]$ and $1 \leq j \leq 2^{p+1}$ there are bounded operators $\mathbb{A}_{\tau,j} \in \mathcal{L}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}))$ such that

$$\|\Pi_j^p \partial_f \Phi_{1,\tau}^*[f] - \mathbb{A}_{\tau,j}[\Pi_j^p f]\|_{1+\alpha} \leq \mu \|\Pi_j^p f\|_{2+\alpha} + K_2 \|f\|_{2+\alpha'} \quad (5.10)$$

for all $f \in h^{2+\alpha}(\mathbb{S})$. The operators $\mathbb{A}_{\tau,j}$ are defined by the formula

$$\mathbb{A}_{\tau,j}[f] := -\frac{\tau k}{\mu_-} \left(\frac{2f'_0 \operatorname{tr}_0 \partial_y v_-^*}{f_* - d} - \operatorname{tr}_0 \partial_x v_-^* \right) \Big|_{x_j^p} f' - \frac{k}{\mu_-} \left(\frac{1 + f_*'^2}{f_* - d} \Big|_{x_j^p} \operatorname{tr}_0 \partial_y w_{-\tau}^{\pi,j}[f] - f'_0(x_j^p) \operatorname{tr}_0 \partial_x w_{-\tau}^{\pi,j}[f] \right), \quad (5.11)$$

where $(w_{+\tau}^{\pi,j}[f], w_{-\tau}^{\pi,j}[f])$ denotes the solution to the problem

$$\left\{ \begin{array}{ll} \mathcal{A}_{0,j}^{\pi}(f_*, h_*) w_{+\tau}^{\pi,j}[f] = \tau A_+ f'' & \text{in } \Omega_+, \\ \mathcal{A}_{0,j}^{\pi}(f_*) w_{-\tau}^{\pi,j}[f] = \tau A_- f'' & \text{in } \Omega_-, \\ \mathcal{B}_j(f_*, h_*) w_{+\tau}^{\pi,j}[f] - \mathcal{B}_j(f_*) w_{-\tau}^{\pi,j}[f] = \tau B f' & \text{on } \Gamma_0, \\ w_{+\tau}^{\pi,j}[f] - w_{-\tau}^{\pi,j}[f] = -[\Delta_\rho + (1 - \tau)\Delta_A]f & \text{on } \Gamma_0, \\ w_{+\tau}^{\pi,j}[f] = 0 & \text{on } \Gamma_1, \\ w_{-\tau}^{\pi,j}[f] = 0 & \text{on } \Gamma_{-1}, \end{array} \right. \quad (5.12)$$

and

$$\begin{aligned} A_+ &:= \frac{\operatorname{tr}_0 \partial_y v_+^*}{h_* - f_*} \Big|_{x_j^p}, & A_- &:= \frac{\operatorname{tr}_0 \partial_y v_-^*}{f_* - d} \Big|_{x_j^p}, & \Delta_\rho &:= g(\rho_- - \rho_+), & \Delta_A &:= A_+ - A_-, \\ B &:= \frac{k}{\mu_-} \left(\frac{2f'_* \operatorname{tr}_0 \partial_y v_-^*}{f_* - d} - \operatorname{tr}_0 \partial_x v_-^* \right) \Big|_{x_j^p} - \frac{k}{\mu_+} \left(\frac{2f'_* \operatorname{tr}_0 \partial_y v_+^*}{h_* - f_*} - \operatorname{tr}_0 \partial_x v_+^* \right) \Big|_{x_j^p}. \end{aligned}$$

Moreover, $\mathcal{A}_{0,j}^{\pi}(f_*, h_*)$, $\mathcal{A}_{0,j}^{\pi}(f_*)$, $\mathcal{B}_j(f_*, h_*)$, and $\mathcal{B}_j(f_*)$ are the operators obtained from $\mathcal{A}_0^{\pi}(f_*, h_*)$, $\mathcal{A}_0^{\pi}(f_*)$, $\mathcal{B}(f_*, h_*)$, and $\mathcal{B}(f_*)$, respectively, when evaluating their coefficients at x_j^p .

Proof. Let $\alpha' \in (0, \alpha)$ and $\mu > 0$ be fixed. Given $p \geq 3$, a p -partition of unity $\{\Pi_j^p\}_{1 \leq j \leq 2^{p+1}}$, and $\tau \in [0, 1]$ we decompose the difference $\Pi_j^p \partial_f \Phi_{1,\tau}^*[f] - \mathbb{A}_{\tau,j}[\Pi_j^p f] = T_1 + T_2 + T_3$ by setting

$$\begin{aligned} T_1 &:= \frac{\tau k}{\mu_-} \left(\frac{2f'_* \operatorname{tr}_0 \partial_y v_-^*}{f_* - d} - \operatorname{tr}_0 \partial_x v_-^* \right) \Big|_{x_j^p} (\Pi_j^p f)' - \frac{\tau k}{\mu_-} \left(\frac{2f'_* \operatorname{tr}_0 \partial_y v_-^*}{f_* - d} - \operatorname{tr}_0 \partial_x v_-^* \right) \Pi_j^p f', \\ T_2 &:= \frac{k}{\mu_-} \frac{1 + f_*'^2}{f_* - d} \Big|_{x_j^p} \operatorname{tr}_0 \partial_y w_{-\tau}^{\pi,j}[\Pi_j^p f] - \frac{k}{\mu_-} \frac{1 + f_*'^2}{f_* - d} \Pi_j^p \operatorname{tr}_0 \partial_y w_{-\tau}^{\pi,j}[f], \\ T_3 &:= \frac{k}{\mu_-} f'_* \Pi_j^p \operatorname{tr}_0 \partial_x w_{-\tau}^{\pi,j}[f] - \frac{k}{\mu_-} f'_*(x_j^p) \operatorname{tr}_0 \partial_x w_{-\tau}^{\pi,j}[\Pi_j^p f] \end{aligned}$$

and estimate each of these terms separately. In the following we shall denote constants which are independent of p (and, of course, of $f \in h^{2+\alpha}(\mathbb{S})$, $\tau \in [0, 1]$, and $j \in \{1, \dots, 2^{p+1}\}$) by C .

The estimate for T_1 . Noticing that $\|\Pi_j^p f''\|_\alpha \leq \|(\Pi_j^p f)''\|_\alpha + K\|f\|_2$ and using the fact that $\chi_j^p \Pi_j^p = \Pi_j^p$, we have

$$\begin{aligned} \|T_1\|_{1+\alpha} &\leq K\|f\|_{1+\alpha} + C \left\| \chi_j^p \left[\left(\frac{2f'_* \operatorname{tr}_0 \partial_y v_-^*}{f_* - d} - \operatorname{tr}_0 \partial_x v_-^* \right) \Big|_{x_j^p} - \left(\frac{2f'_* \operatorname{tr}_0 \partial_y v_-^*}{f_* - d} - \operatorname{tr}_0 \partial_x v_-^* \right) \right] \Pi_j^p f' \right\|_{1+\alpha} \\ &\leq C \left[\left\| \chi_j^p \left(\frac{f'_* \operatorname{tr}_0 \partial_y v_-^*}{f_* - d} \Big|_{x_j^p} - \frac{f'_* \operatorname{tr}_0 \partial_y v_-^*}{f_* - d} \right) \right\|_0 + \|\chi_j^p (\operatorname{tr}_0 \partial_x v_-^* - \partial_x v_-^*(x_j^p, 0))\|_0 \right] \|\Pi_j^p f\|_{2+\alpha} \\ &\quad + K\|f\|_2. \end{aligned}$$

Choosing p sufficiently large and using the uniform continuity of the functions in the square brackets, we are led to the estimate

$$\|T_1\|_{1+\alpha} \leq (\mu/3) \|\Pi_j^p f\|_{2+\alpha} + K\|f\|_2. \quad (5.13)$$

The estimate for T_2 and T_3 . The terms T_2 and T_3 are estimated in a similar way. However, due to the nonlocal character of these expressions, the arguments are more involved than in the previous step. It is easy to see that

$$\begin{aligned} \|T_3\|_{1+\alpha} &\leq C \|f'_* \Pi_j^p \operatorname{tr}_0 \partial_x w_{-\tau}^{\pi,j}[f] - f'_*(x_j^p) \operatorname{tr}_0 \partial_x w_{-\tau}^{\pi,j}[\Pi_j^p f]\|_{1+\alpha} \\ &\leq C \|\chi_j^p f'_* \operatorname{tr}_0 \partial_x (\Pi_j^p w_{-\tau}^{\pi,j}[f]) - f'_*(x_j^p) \operatorname{tr}_0 \partial_x w_{-\tau}^{\pi,j}[\Pi_j^p f]\|_{1+\alpha} + K\|f\|_{2+\alpha'} \\ &\leq C \|\chi_j^p f'_* \operatorname{tr}_0 \partial_x (\Pi_j^p w_{-\tau}^{\pi,j}[f]) - \chi_j^p f'_*(x_j^p) \operatorname{tr}_0 \partial_x w_{-\tau}^{\pi,j}[\Pi_j^p f]\|_{1+\alpha} \\ &\quad + C \|(1 - \chi_j^p) f'_*(x_j^p) \operatorname{tr}_0 \partial_x w_{-\tau}^{\pi,j}[\Pi_j^p f]\|_{1+\alpha} + K\|f\|_{2+\alpha'} \\ &=: T_{31} + T_{32} + K\|f\|_{2+\alpha'}, \end{aligned} \quad (5.14)$$

where we used once more the fact that $\chi_j^p \Pi_j^p = \Pi_j^p$. We first note that

$$T_{32} \leq C \|(1 - \chi_j^p) \operatorname{tr}_0 \partial_x w_{-\tau}^{\pi,j}[\Pi_j^p f]\|_{1+\alpha} \leq K\|f\|_{2+\alpha'} + C \|(1 - \chi_j^p) w_{-\tau}^{\pi,j}[\Pi_j^p f]\|_{2+\alpha}^{\Omega_-}.$$

The pair $(u_+, u_-) := (1 - \chi_j^p)(w_{+\tau}^{\pi,j}[\Pi_j^p f], w_{-\tau}^{\pi,j}[\Pi_j^p f])$ is the solution to

$$\left\{ \begin{array}{ll} \mathcal{A}_{0,j}^\pi(f_*, h_*) u_+ = \tau A_+ (1 - \chi_j^p) (\Pi_j^p f)'' - (\chi_j^p)'' w_{+\tau}^{\pi,j}[\Pi_j^p f] \\ \quad - 2(\chi_j^p)' \partial_x w_{+\tau}^{\pi,j}[\Pi_j^p f] + \frac{2f'_*}{h_* - f_*} \Big|_{x_j^p} (\chi_j^p)' \partial_y w_{+\tau}^{\pi,j}[\Pi_j^p f] & \text{in } \Omega_+, \\ \mathcal{A}_{0,j}^\pi(f_*) u_- = \tau A_- (1 - \chi_j^p) (\Pi_j^p f)'' - (\chi_j^p)'' w_{-\tau}^{\pi,j}[\Pi_j^p f] \\ \quad - 2(\chi_j^p)' \partial_x w_{-\tau}^{\pi,j}[\Pi_j^p f] + \frac{2f'_*}{f_* - d} \Big|_{x_j^p} (\chi_j^p)' \partial_y w_{-\tau}^{\pi,j}[\Pi_j^p f] & \text{in } \Omega_-, \\ \mathcal{B}_j(f_*, h_*) u_+ - \mathcal{B}_j(f_*) u_- = \tau B (1 - \chi_j^p) (\Pi_j^p f)' + k \mu_+^{-1} (\chi_j^p)' f'_*(x_j^p) \operatorname{tr}_0 w_{+\tau}^{\pi,j}[\Pi_j^p f] \\ \quad - k \mu_-^{-1} (\chi_j^p)' f'_*(x_j^p) \operatorname{tr}_0 w_{-\tau}^{\pi,j}[\Pi_j^p f] & \text{on } \Gamma_0, \\ u_+ - u_- = -[\Delta_\rho + (1 - \tau) \Delta_A] (1 - \chi_j^p) \Pi_j^p f & \text{on } \Gamma_0, \\ u_+ = 0 & \text{on } \Gamma_1, \\ u_- = 0 & \text{on } \Gamma_{-1}, \end{array} \right.$$

and, because of $(1 - \chi_j^p) \Pi_j^p = 0$, we end up with

$$T_{32} \leq K\|f\|_{2+\alpha'} + C \|u_-\|_{2+\alpha}^{\Omega_-} \leq K\|f\|_{2+\alpha'}. \quad (5.15)$$

The remaining term T_{31} can be estimated as

$$\begin{aligned} T_{31} &\leq C \|\chi_j^p (f'_* - f'_*(x_j^p)) \operatorname{tr}_0 \partial_x w_{-\tau}^{\pi,j} [\Pi_j^p f]\|_{1+\alpha} + C \|\chi_j^p \operatorname{tr}_0 \partial_x (\Pi_j^p w_{-\tau}^\pi [f] - w_{-\tau}^{\pi,j} [\Pi_j^p f])\|_{1+\alpha} \\ &\leq (\mu/6) \|\Pi_j^p f\|_{2+\alpha} + K \|f\|_{2+\alpha'} + C \|\chi_j^p \operatorname{tr}_0 \partial_x (\Pi_j^p w_{-\tau}^\pi [f] - w_{-\tau}^{\pi,j} [\Pi_j^p f])\|_{1+\alpha} \end{aligned} \quad (5.16)$$

if p is sufficiently large. We are left to estimate the last term in (5.16). We note that

$$\|\chi_j^p \operatorname{tr}_0 \partial_x (\Pi_j^p w_{-\tau}^\pi [f] - w_{-\tau}^{\pi,j} [\Pi_j^p f])\|_{1+\alpha} \leq K \|f\|_{2+\alpha'} + \|\chi_j^p (w_{-\tau}^{\pi,j} [\Pi_j^p f] - \Pi_j^p w_{-\tau}^\pi [f])\|_{2+\alpha}^{\Omega_-}. \quad (5.17)$$

We now infer from the fact that the pair

$$(z_+, z_-) := \chi_j^p ((w_{+\tau}^{\pi,j} [\Pi_j^p f], w_{-\tau}^{\pi,j} [\Pi_j^p f]) - \Pi_j^p (w_{+\tau}^\pi [f], w_{-\tau}^\pi [f]))$$

solves the diffraction problem

$$\left\{ \begin{aligned} \mathcal{A}_0^\pi(f_*, h_*) z_+ &= (\mathcal{A}_0^\pi(f_*, h_*) - \mathcal{A}_{0,j}^\pi(f_*, h_*)) [\chi_j^p w_{+\tau}^{\pi,j} [\Pi_j^p f]] \\ &\quad + \tau A_+ \chi_j^p (\Pi_j^p f)'' - \frac{\tau \operatorname{tr}_0 \partial_y v_+^*}{h_* - f_*} \Pi_j^p f'' \\ &\quad + (\chi_j^p)'' w_{+\tau}^{\pi,j} [\Pi_j^p f] - (\Pi_j^p)'' w_{+\tau}^\pi [f] \\ &\quad + 2(\chi_j^p)' \partial_x w_{+\tau}^{\pi,j} [\Pi_j^p f] - 2(\Pi_j^p)' \partial_x w_{+\tau}^\pi [f] \\ &\quad + \frac{2f'_*}{h_* - f_*} (\Pi_j^p)' \partial_y w_{+\tau}^\pi [f] - \frac{2f'_*}{h_* - f_*} \Big|_{x_j^p} (\chi_j^p)' \partial_y w_{+\tau}^{\pi,j} [\Pi_j^p f] \quad \text{in } \Omega_+, \\ \mathcal{A}_0^\pi(f_*) z_- &= (\mathcal{A}_0^\pi(f_*) - \mathcal{A}_{0,j}^\pi(f_*)) [\chi_j^p w_{-\tau}^{\pi,j} [\Pi_j^p f]] \\ &\quad + \tau A_- \chi_j^p (\Pi_j^p f)'' - \frac{\tau \operatorname{tr}_0 \partial_y v_-^*}{f_* - d} \Pi_j^p f'' \\ &\quad + (\chi_j^p)'' w_{-\tau}^{\pi,j} [\Pi_j^p f] - (\Pi_j^p)'' w_{-\tau}^\pi [f] \\ &\quad + 2(\chi_j^p)' \partial_x w_{-\tau}^{\pi,j} [\Pi_j^p f] - 2(\Pi_j^p)' \partial_x w_{-\tau}^\pi [f] \\ &\quad + \frac{2f'_*}{f_* - d} (\Pi_j^p)' \partial_y w_{-\tau}^\pi [f] - \frac{2f'_*}{f_* - d} \Big|_{x_j^p} (\chi_j^p)' \partial_y w_{-\tau}^{\pi,j} [\Pi_j^p f] \quad \text{in } \Omega_-, \\ \mathcal{B}(f_*, h_*) z_+ - \mathcal{B}(f_*) z_- &= (\mathcal{B}(f_*, h_*) - \mathcal{B}_j(f_*, h_*)) [\chi_j^p w_{+\tau}^{\pi,j} [\Pi_j^p f]] \\ &\quad - (\mathcal{B}(f_*) - \mathcal{B}_j(f_*)) [\chi_j^p w_{-\tau}^{\pi,j} [\Pi_j^p f]] + \tau B \chi_j^p (\Pi_j^p f)' \\ &\quad + \tau \Pi_j^p \partial_f \mathcal{B}^\pi(f_*, h_*) [f] v_+^* - \tau \Pi_j^p \partial_f \mathcal{B}^\pi(f_*) [f] v_-^* \\ &\quad - k \mu_+^{-1} ((\chi_j^p)' f'_*(x_j^p) \operatorname{tr}_0 w_{+\tau}^{\pi,j} [\Pi_j^p f] - (\Pi_j^p)' f'_* \operatorname{tr}_0 w_{+\tau}^\pi [f]) \\ &\quad + k \mu_-^{-1} ((\chi_j^p)' f'_*(x_j^p) \operatorname{tr}_0 w_{-\tau}^{\pi,j} [\Pi_j^p f] - (\Pi_j^p)' f'_* \operatorname{tr}_0 w_{-\tau}^\pi [f]) \quad \text{on } \Gamma_0, \\ z_+ - z_- &= (1 - \tau) \chi_j^p \left(\frac{\tau \operatorname{tr}_0 \partial_y v_+^*}{h_* - f_*} - \frac{\tau \operatorname{tr}_0 \partial_y v_-^*}{f_* - d} - \Delta_A \right) \Pi_j^p f \quad \text{on } \Gamma_0, \\ z_+ &= 0 \quad \text{on } \Gamma_1, \\ z_- &= 0 \quad \text{on } \Gamma_{-1} \end{aligned} \right.$$

and the Schauder estimate (3.6) that

$$\begin{aligned}
\|z_-\|_{2+\alpha}^{\Omega_-} \leq & C \left(\|(\mathcal{A}_0^\pi(f_*, h_*) - \mathcal{A}_{0,j}^\pi(f_*, h_*)) [\chi_j^p w_{+\tau}^{\pi,j} [\Pi_j^p f]]\|_\alpha + \left\| \chi_j^p \left(\frac{\text{tr}_0 \partial_y v_+^*}{h_* - f_*} - A_+ \right) \right\|_\alpha \|\Pi_j^p f''\|_0 \right. \\
& + \|(\mathcal{A}_0^\pi(f_*) - \mathcal{A}_{0,j}^\pi(f_*)) [\chi_j^p w_{-\tau}^{\pi,j} [\Pi_j^p f]]\|_\alpha + \left\| \chi_j^p \left(\frac{\text{tr}_0 \partial_y v_-^*}{f_* - d} - A_- \right) \right\|_\alpha \|\Pi_j^p f''\|_0 \\
& + \|(\mathcal{B}(f_*, h_*) - \mathcal{B}_j(f_*, h_*)) [\chi_j^p w_{+\tau}^{\pi,j} [\Pi_j^p f]]\|_{1+\alpha} \\
& + \|(\mathcal{B}(f_*) - \mathcal{B}_j(f_*)) [\chi_j^p w_{-\tau}^{\pi,j} [\Pi_j^p f]]\|_{1+\alpha} \\
& + \|\tau B \chi_j^p (\Pi_j^p f)' + \tau \Pi_j^p \partial_f \mathcal{B}^\pi(f_*, h_*) [f] v_+^* - \tau \Pi_j^p \partial_f \mathcal{B}^\pi(f_*) [f] v_-^*\|_{1+\alpha} \\
& + \left\| \chi_j^p \left(\frac{\tau \text{tr}_0 \partial_y v_+^*}{h_* - f_*} - \frac{\tau \text{tr}_0 \partial_y v_-^*}{f_* - d} - \Delta_A \right) \right\|_0 \|\Pi_j^p f\|_{2+\alpha} \Big) \\
& + K \|f\|_{2+\alpha'}.
\end{aligned}$$

Using the same arguments as when estimating T_1 , we find for p sufficiently large

$$\|\chi_j^p (w_{-\tau}^{\pi,j} [\Pi_j^p f] - \Pi_j^p w_{-\tau}^\pi [f])\|_{2+\alpha}^{\Omega_-} = \|z_-\|_{2+\alpha}^{\Omega_-} \leq (\mu/6) \|\Pi_j^p f\|_{2+\alpha} + K \|f\|_{2+\alpha'},$$

which yields, together with (5.15)-(5.17), that

$$\|T_3\|_{1+\alpha} \leq (\mu/3) \|\Pi_j^p f\|_{2+\alpha} + K \|f\|_{2+\alpha'}. \quad (5.18)$$

Similar arguments show that

$$\|T_2\|_{1+\alpha} \leq (\mu/3) \|\Pi_j^p f\|_{2+\alpha} + K \|f\|_{2+\alpha'}. \quad (5.19)$$

Gathering (5.13), (5.18), and (5.19), we have established the desired estimate (5.10). \square

Fourier analysis: The symbol of $\mathbb{A}_{\tau,j}$. In this step we use Fourier analysis arguments and ODE techniques to represent the operators $\mathbb{A}_{\tau,j}$ introduced in (5.11) as Fourier multipliers. Subsequently we will use a Marcinkiewicz type Fourier multiplier theorem to prove that all these Fourier multipliers $\mathbb{A}_{\tau,j}$ are generators of strongly continuous and analytic semigroups, cf. Lemma 5.6, provided the more dense fluid lies below the less dense one and under certain conditions on the initial data.

We fix now $\tau \in [0, 1]$ and $p, j \in \mathbb{N}$ with $p \geq 3$ and $1 \leq j \leq 2^{p+1}$ arbitrary. Given $f \in h^{2+\alpha}(\mathbb{S})$, we consider its Fourier expansion

$$f = \sum_{m \in \mathbb{Z}} f_m e^{imx}$$

and look for the corresponding expansion of $\mathbb{A}_{\tau,j}[f]$. In view of (5.11), we needed to determine the expansions for $\text{tr}_0 \nabla(w_{+\tau}^{\pi,j}[f], w_{-\tau}^{\pi,j}[f])$. Hence, we let

$$(w_{+\tau}^{\pi,j}[f], w_{-\tau}^{\pi,j}[f]) = \sum_{m \in \mathbb{Z}} (A_m^+, A_m^-) f_m e^{imx} \quad (5.20)$$

with functions $(A_m^+, A_m^-) = (A_m^+, A_m^-)(y)$ still to be determined. Recalling (5.3) and the formulae in Appendix A, we use the summation convention and write

$$\mathcal{A}_{0,j}^\pi(f_*, h_*) = c_{pl}^+ \partial_{pl}, \quad \mathcal{A}_{0,j}^\pi(f_*) = c_{pl}^- \partial_{pl}, \quad \mathcal{B}_j(f_*, h_*) = \beta_l^+ \text{tr}_0 \partial_l, \quad \mathcal{B}_j(f_*) = \beta_l^- \text{tr}_0 \partial_l,$$

and we set

$$a_\pm := c_{12}^\pm / c_{22}^\pm, \quad b_\pm := 1/c_{22}^\pm, \quad D_\pm := \sqrt{b_\pm - (a_\pm)^2}$$

noticing that $b_{\pm} - (a_{\pm})^2 > 0$. From (5.20) and (5.12) we deduce that for fixed $m \in \mathbb{Z}$, the pair (A_m^+, A_m^-) solves the following problem

$$\left\{ \begin{array}{ll} (A_m^+)'' + i2ma_+(A_m^+) - b_+m^2A_m^+ = -\tau b_+A_+m^2 & \text{in } 0 < y < 1, \\ (A_m^-)'' + i2ma_-(A_m^-) - b_-m^2A_m^- = -\tau b_-A_-m^2 & \text{in } -1 < y < 0, \\ im(\beta_1^+A_m^+(0) - \beta_1^-A_m^-(0)) + \beta_2^+(A_m^+)'(0) - \beta_2^-(A_m^-)'(0) = i\tau mB, \\ A_m^+(0) - A_m^-(0) = -[\Delta_\rho + (1 - \tau)\Delta_A], \\ A_m^+(1) = 0, \\ A_m^-(-1) = 0, \end{array} \right. \quad (5.21)$$

with constants $A_{\pm}, B, \Delta_\rho, \Delta_A$ defined in Theorem 5.5. We now set

$$\begin{aligned} \cos_{\pm}(y) &:= \cos(a_{\pm}my), & \sin_{\pm}(y) &:= \sin(a_{\pm}my), \\ \cosh_{\pm}(y) &:= \cosh(D_{\pm}my), & \sinh_{\pm}(y) &:= \sinh(D_{\pm}my). \end{aligned}$$

With this notation, it is easy to verify that the general solutions to the first two equations of (5.21) are given by the formula

$$A_m^{\pm} := u_m^{\pm} + iv_m^{\pm} \quad (5.22)$$

with real parts

$$\begin{aligned} u_m^{\pm}(y) = & \xi_1^{\pm} \left(\cos_{\pm}(y) \cosh_{\pm}(y) + \frac{a_{\pm}}{D_{\pm}} \sin_{\pm}(y) \sinh_{\pm}(y) \right) + \frac{\xi_2^{\pm}}{D_{\pm}m} \cos_{\pm}(y) \sinh_{\pm}(y) \\ & + \xi_3^{\pm} \left(\sin_{\pm}(y) \cosh_{\pm}(y) - \frac{a_{\pm}}{D_{\pm}} \cos_{\pm}(y) \sinh_{\pm}(y) \right) + \frac{\xi_4^{\pm}}{D_{\pm}m} \sin_{\pm}(y) \sinh_{\pm}(y) + \tau A_{\pm} \end{aligned}$$

and imaginary parts

$$\begin{aligned} v_m^{\pm}(y) = & \xi_1^{\pm} \left(-\sin_{\pm}(y) \cosh_{\pm}(y) + \frac{a_{\pm}}{D_{\pm}} \cos_{\pm}(y) \sinh_{\pm}(y) \right) - \frac{\xi_2^{\pm}}{D_{\pm}m} \sin_{\pm}(y) \sinh_{\pm}(y) \\ & + \xi_3^{\pm} \left(\cos_{\pm}(y) \cosh_{\pm}(y) + \frac{a_{\pm}}{D_{\pm}} \sin_{\pm}(y) \sinh_{\pm}(y) \right) + \frac{\xi_4^{\pm}}{D_{\pm}m} \cos_{\pm}(y) \sinh_{\pm}(y). \end{aligned}$$

The real constants $\{\xi_i^{\pm} : 1 \leq i \leq 4\}$ are to be determined such that the last four equations of (5.21) are also satisfied by (A_m^+, A_m^-) . It can be shown by explicit, but tedious computations that such constants can be uniquely determined to obtain (A_m^+, A_m^-) . Since

$$\text{tr}_0 \partial_y w_{-\tau}^{\pi,j}[f] = \sum_{m \in \mathbb{Z}} (\xi_2^- + i\xi_4^-) f_m e^{imx}, \quad \text{tr}_0 \partial_x w_{-\tau}^{\pi,j}[f] = \sum_{m \in \mathbb{Z}} im(\xi_1 + \tau A_- + i\xi_3) f_m e^{imx},$$

and using the definition (5.11) together with the explicit expressions for $\{\xi_i^{\pm} : 1 \leq i \leq 4\}$, it follows that $\mathbb{A}_{\tau,j}$ is the Fourier multiplier

$$\mathbb{A}_{\tau,j} \left[\sum_m f_m e^{imx} \right] = \sum_m \lambda_m f_m e^{imx}$$

with symbol $(\lambda_m)_{m \in \mathbb{Z}}$ defined by

$$\operatorname{Re} \lambda_m := - \left(\frac{\tanh(D_+ m)}{\beta_2^+ D_+ m} + \frac{\tanh(D_- m)}{\beta_2^- D_- m} \right)^{-1} \left[\Delta_\rho + \Delta_A + \frac{\tau A_- \cos(D_- m)}{\cosh(D_- m)} - \frac{\tau A_+ \cos(D_+ m)}{\cosh(D_+ m)} \right], \quad (5.23)$$

$$\begin{aligned} \operatorname{Im} \lambda_m := & \frac{\tau k}{\mu_-} \left(\frac{a_- \partial_y v_-^*(x_j^p, 0)}{a_-^2 + D_-^2} + \partial_x v_-^*(x_j^p, 0) \right) m \\ & + \tau \left(\frac{\tanh(D_+ m)}{\beta_2^+ D_+ m} + \frac{\tanh(D_- m)}{\beta_2^- D_- m} \right)^{-1} \left[\frac{A_- \sin(D_- m)}{\cosh(D_- m)} + \frac{A_+ \sin(D_+ m)}{\cosh(D_+ m)} \right] \\ & - \tau \left(\frac{\tanh(D_+ m)}{\beta_2^+ D_+ m} + \frac{\tanh(D_- m)}{\beta_2^- D_- m} \right)^{-1} \frac{\tanh(D_+ m)}{\beta_2^+ D_+} [\beta_1^+ A_+ - \beta_1^- A_- - B]. \end{aligned} \quad (5.24)$$

We refrain from presenting here the detailed computations that are used to determine the constants $\{\xi_i^\pm : 1 \leq i \leq 4\}$, (5.23), and (5.24), as they are quite long and only the outcome, that is, the explicit formula for the symbol $(\lambda_m)_{m \in \mathbb{Z}}$, is of importance for the further analysis.

We are now in the position to prove that the operators $\mathbb{A}_{\tau,j}$ are generators of strongly continuous and analytic semigroups. To this end we will use a Marcinkiewicz type Fourier multiplier theorem [29, Theorem 2.1], which generalizes a result from [4] (see also [19] for a similar result) and states that a Fourier multiplier

$$\sum_{m \in \mathbb{Z}} f_m e^{imx} \mapsto \sum_{m \in \mathbb{Z}} \Lambda_m f_m e^{imx}$$

belongs to $\mathcal{L}(C^{r+\alpha}(\mathbb{S}), C^{s+\alpha}(\mathbb{S}))$, $r, s \in \mathbb{N}$ and $\alpha \in (0, 1)$, provided that

$$s_1 := \sup_{m \in \mathbb{Z} \setminus \{0\}} |m|^{s-r} |\Lambda_m| < \infty \quad \text{and} \quad s_2 := \sup_{m \in \mathbb{Z} \setminus \{0\}} |m|^{s-r+1} |\Lambda_{m+1} - \Lambda_m| < \infty.$$

Additionally, the norm of the Fourier multiplier as a bounded linear operator from $C^{r+\alpha}(\mathbb{S})$ to $C^{s+\alpha}(\mathbb{S})$ can be bounded in terms of the constants s_1 and s_2 alone. Bearing this result in mind we can derive the following lemma.

Lemma 5.6. *Let $\sigma > 0$ be such that $(f_0, h_0) \in S_\sigma$. Then, there exist constants $\kappa_1 \geq 1$ and $\omega_1 > 0$ depending only on σ , such that for any $(f_*, h_*) \in (h^{3+\alpha}(\mathbb{S}))^2 \cap S_\sigma$, any p -partition of unity $\{\Pi_j^p\}_{1 \leq j \leq 2^{p+1}}$ with $p \geq 3$ and any $\tau \in [0, 1]$, the operators $\mathbb{A}_{\tau,j}$, $1 \leq j \leq 2^{p+1}$, defined by (5.11) satisfy*

$$\lambda - \mathbb{A}_{\tau,j} \in \operatorname{Isom}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})), \quad (5.25)$$

$$\kappa_1 \|(\lambda - \mathbb{A}_{\tau,j})[f]\|_{1+\alpha} \geq |\lambda| \cdot \|f\|_{1+\alpha} + \|f\|_{2+\alpha} \quad (5.26)$$

for all $f \in h^{2+\alpha}(\mathbb{S})$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geq \omega_1$.

Proof. We write

$$\operatorname{Re} \lambda_m = -\mu_m - \nu_m,$$

where

$$\begin{aligned}\nu_m &:= \tau \left(\frac{\tanh(D_+ m)}{\beta_2^+ D_+ m} + \frac{\tanh(D_- m)}{\beta_2^- D_- m} \right)^{-1} \left[\frac{A_- \cos(D_- m)}{\cosh(D_- m)} - \frac{A_+ \cos(D_+ m)}{\cosh(D_+ m)} \right], \\ \mu_m &:= \left(\frac{\tanh(D_+ m)}{\beta_2^+ D_+ m} + \frac{\tanh(D_- m)}{\beta_2^- D_- m} \right)^{-1} (\Delta_\rho + \Delta_A).\end{aligned}$$

Recalling the definition of S_σ , there exists a constant $C_0 = C_0(\sigma) > 1$ such that

$$C_0^{-1} \leq \Delta_\rho + \Delta_A \leq C_0. \quad (5.27)$$

In fact, it is not difficult to see that we can choose C_0 large enough to guarantee that

$$\begin{aligned}C_0^{-1} \leq \frac{\mu_m}{|m|} &\leq C_0, \quad m \in \mathbb{Z} \setminus \{0\}, \tau \in [0, 1], \\ \sup_{m \in \mathbb{Z} \setminus \{0\}} \sup_{\tau \in [0, 1]} \left(|\nu_m| + \frac{|\operatorname{Re} \lambda_m| + |\operatorname{Im} \lambda_m|}{|m|} \right) &\leq C_0, \\ \sup_{m \in \mathbb{Z} \setminus \{0\}} \sup_{\tau \in [0, 1]} \left(|\operatorname{Re} \lambda_{m+1} - \operatorname{Re} \lambda_m| + |\operatorname{Im} \lambda_{m+1} - \operatorname{Im} \lambda_m| \right) &\leq C_0.\end{aligned} \quad (5.28)$$

Let now $\lambda = \zeta_1 + i\zeta_2 \in \mathbb{C}$ be given such that $\operatorname{Re} \lambda = \zeta_1 \geq \omega_1 := 2C_0$. Since $\zeta_1 - \operatorname{Re} \lambda_m \geq C_0$, the (formal) inverse $(\lambda - \mathbb{A}_{\tau,j})^{-1}$ of $\lambda - \mathbb{A}_{\tau,j}$ is the Fourier multiplier

$$\sum_{m \in \mathbb{Z}} f_m e^{imx} \mapsto \sum_{m \in \mathbb{Z}} \Lambda_m f_m e^{imx}$$

with symbol

$$\Lambda_m := (\lambda - \lambda_m)^{-1} = \frac{\zeta_1 - \operatorname{Re} \lambda_m - i(\zeta_2 - \operatorname{Im} \lambda_m)}{(\zeta_1 - \operatorname{Re} \lambda_m)^2 + (\zeta_2 - \operatorname{Im} \lambda_m)^2}.$$

Step 1. We first prove that $(\lambda - \mathbb{A}_{\tau,j})^{-1}$ belongs to $\mathcal{L}(C^{1+\alpha}(\mathbb{S}), C^{2+\alpha}(\mathbb{S}))$, the norm being independent of $p \geq 3, j \in \{1, \dots, 2^{p+1}\}, \tau \in [0, 1]$, and $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geq \omega_1$. More precisely, we show that¹

$$\sup_{\zeta_1 \geq \omega_1} \sup_{x_j^p \in \mathbb{S}} \sup_{\tau \in [0, 1]} \sup_{m \in \mathbb{Z}^*} (|m \Lambda_m| + |m|^2 |\Lambda_{m+1} - \Lambda_m|) < \infty. \quad (5.29)$$

Note that (5.28) implies

$$\begin{aligned}|\Lambda_0| &\leq C_0, \\ |m|^2 |\Lambda_m|^2 &= \frac{m^2}{(\zeta_1 - \operatorname{Re} \lambda_m)^2 + (\zeta_2 - \operatorname{Im} \lambda_m)^2} \leq \frac{m^2}{(\zeta_1 - \operatorname{Re} \lambda_m)^2} \leq \frac{C_0^2 m^2}{m^2 + C_0^4} \leq C_0^2.\end{aligned} \quad (5.30)$$

Setting $R_m := \zeta_1 - \operatorname{Re} \lambda_m$ and $I_m := \zeta_2 - \operatorname{Im} \lambda_m$, we have $R_m > 0$ and

$$|\Lambda_{m+1} - \Lambda_m| \leq T_1 + T_2 + T_3 + T_4,$$

¹The operator $\mathbb{A}_{\tau,j}$ depends on the partition $\{\Pi_j^p\}_{1 \leq j \leq 2^{p+1}}$ through the middle point x_j^p of the interval I_j^p only, and therefore $\mathbb{A}_{\tau,j}$ makes sense when replacing x_j^p by any $x \in \mathbb{S}$. In view of this fact, when taking the supremum over $x_j^p \in \mathbb{S}$ in (5.29) we consider a larger set than when taking the supremum over all middle point x_j^p and all p -partitions.

where

$$\begin{aligned} T_1 &:= \frac{R_m R_{m+1} |R_{m+1} - R_m|}{(R_m^2 + I_m^2)(R_{m+1}^2 + I_{m+1}^2)}, & T_2 &:= \frac{|R_m I_{m+1}^2 - R_{m+1} I_m^2|}{(R_m^2 + I_m^2)(R_{m+1}^2 + I_{m+1}^2)}, \\ T_3 &:= \frac{|I_m I_{m+1}| |I_{m+1} - I_m|}{(R_m^2 + I_m^2)(R_{m+1}^2 + I_{m+1}^2)}, & T_4 &:= \frac{|I_m R_{m+1}^2 - I_{m+1} R_m^2|}{(R_m^2 + I_m^2)(R_{m+1}^2 + I_{m+1}^2)}. \end{aligned}$$

The estimates for $m^2 T_3$ and $m^2 T_4$ are similar to those for $m^2 T_1$ and $m^2 T_2$, and therefore we shall present only those for the latter. Recalling (5.28) and (5.30), we get

$$m^2 T_1 \leq m^2 |\Lambda_m| \cdot |\Lambda_{m+1}| \cdot |R_{m+1} - R_m| \leq 2C_0^3$$

and

$$\begin{aligned} m^2 T_2 &\leq m^2 \frac{|R_m|(|I_{m+1}| + |I_m|)|I_{m+1} - I_m| + |I_m^2||R_{m+1} - R_m|}{(R_m^2 + I_m^2)(R_{m+1}^2 + I_{m+1}^2)} \leq C_0 m^2 |\Lambda_{m+1}| (|\Lambda_m| + 2|\Lambda_{m+1}|) \\ &\leq 10C_0^3. \end{aligned}$$

Proceeding similarly with $m^2 T_3$ and $m^2 T_4$, we arrive at (5.29). In view of [29, Theorem 2.1] we additionally know that $(\lambda - \mathbb{A}_{\tau,j})^{-1} \in \mathcal{L}(C^{n+\alpha}(\mathbb{S}), C^{n+1+\alpha}(\mathbb{S}))$ for all $n \in \mathbb{N}$, and since the closure of $C^{n+1+\alpha}(\mathbb{S})$ in $C^{n+\alpha}(\mathbb{S})$ is exactly $h^{n+\alpha}(\mathbb{S})$, a density argument leads us to the desired property (5.25). Moreover, our arguments show that there exists a constant κ_1 depending only on C_0 such that

$$\kappa_1 \|(\lambda - \mathbb{A}_{\tau,j})[f]\|_{1+\alpha} \geq \|f\|_{2+\alpha} \quad \text{for all } f \in h^{2+\alpha}(\mathbb{S}) \text{ and } \operatorname{Re} \lambda \geq \omega_1. \quad (5.31)$$

Step 2. We are left to prove that we can choose κ_1 large enough to guarantee that

$$\kappa_1 \|(\lambda - \mathbb{A}_{\tau,j})[f]\|_{1+\alpha} \geq |\lambda| \cdot \|f\|_{1+\alpha} \quad \text{for all } f \in h^{2+\alpha}(\mathbb{S}) \text{ and } \operatorname{Re} \lambda \geq \omega_1.$$

For this it suffices to show that

$$\sup_{\zeta_1 \geq \omega_1} \sup_{x_j^p \in \mathbb{S}} \sup_{\tau \in [0,1]} \sup_{m \in \mathbb{Z}^*} |\lambda| (|\Lambda_m| + |m| \cdot |\Lambda_{m+1} - \Lambda_m|) < \infty. \quad (5.32)$$

Using (5.28), we first note that

$$\frac{\zeta_1^2}{(\zeta_1 - \operatorname{Re} \lambda_m)^2 + (\zeta_2 - \operatorname{Im} \lambda_m)^2} \leq \frac{\zeta_1^2}{(\zeta_1 - \operatorname{Re} \lambda_m)^2} \leq \frac{\zeta_1^2}{(\zeta_1/2)^2} = 4.$$

Additionally, (5.28) and (5.30) lead us to

$$\begin{aligned} \frac{\zeta_2^2}{(\zeta_1 - \operatorname{Re} \lambda_m)^2 + (\zeta_2 - \operatorname{Im} \lambda_m)^2} &\leq \frac{2(\zeta_2 - \operatorname{Im} \lambda_m)^2 + 2(\operatorname{Im} \lambda_m)^2}{(\zeta_1 - \operatorname{Re} \lambda_m)^2 + (\zeta_2 - \operatorname{Im} \lambda_m)^2} \leq 2 \left[1 + \frac{|\operatorname{Im} \lambda_m|^2}{m^2} (m^2 |\Lambda_m|^2) \right] \\ &\leq 2(1 + C_0^4), \end{aligned}$$

and we thus have established that

$$\sup_{\zeta_1 \geq \omega_1} \sup_{x_j^p \in \mathbb{S}} \sup_{\tau \in [0,1]} \sup_{m \in \mathbb{Z}^*} |\lambda| \cdot |\Lambda_m| \leq 2(2 + C_0^2). \quad (5.33)$$

To finish the proof, we recall that $|\Lambda_{m+1} - \Lambda_m| \leq T_1 + T_2 + T_3 + T_4$. Combining (5.28), (5.30), and (5.33), we find that

$$|\lambda m| T_1 \leq |\lambda m| \cdot |\Lambda_m| \cdot |\Lambda_{m+1}| \cdot |R_{m+1} - R_m| \leq 4C_0^2(2 + C_0^2)$$

and

$$|\lambda m|T_2 \leq C_0 |\lambda m| \cdot |\Lambda_{m+1}| (|\Lambda_m| + 2|\Lambda_{m+1}|) \leq 10C_0^2(2 + C_0^2).$$

Proceeding similarly with the remaining terms $|\lambda m|T_3$ and $|\lambda m|T_4$, we obtain (5.32). Together with (5.31) we deduce (5.26), and the proof is completed. \square

With these preliminary results, we can now prove that the assumptions of Lemma 5.2 hold and establish in this way (3.16).

Theorem 5.7. *Let $(f_0, h_0) \in \mathcal{V}$ and $b_0 \in h^{2+\alpha}(\mathbb{S})$ be given such that the first inequality in (2.7) is satisfied. Then*

$$-\partial_f \Phi_1(0, (f_0, h_0)) \in \mathcal{H}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})).$$

Proof. Recalling (5.4) (with $(f_*, h_*) = (f_0, h_0)$) and the perturbation result [2, Theorem I.1.3.1 (ii)], we are left to show that $-\partial_f \Phi_1^\pi(f_0, h_0) \in \mathcal{H}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}))$. To this end, it suffices to verify the assumptions of Lemma 5.2 for some $\sigma > 0$.

Let $\sigma > 0$ be such that $(f_0, h_0) \in S_\sigma$, let κ_1 and ω_1 be the constants found in Lemma 5.6, and pick $\alpha' \in (0, \alpha)$. Given $(f_*, h_*) \in (h^{3+\alpha}(\mathbb{S}))^2 \cap S_\sigma$, Theorem 5.5 implies the existence of a p -partition of unity $\{\Pi_j^p\}_{1 \leq j \leq 2^{p+1}}$ and of a constant K_2 such that

$$\|\Pi_j^p \partial_f \Phi_{1,\tau}^{\pi*}[f] - \mathbb{A}_{\tau,j}[\Pi_j^p f]\|_{1+\alpha} \leq \frac{1}{2\kappa_1} \|\Pi_j^p f\|_{2+\alpha} + K_2 \|f\|_{2+\alpha'} \quad (5.34)$$

for all $f \in h^{2+\alpha}(\mathbb{S})$, $\tau \in [0, 1]$, and $j \in \{1, \dots, 2^{p+1}\}$. In view of (5.26), we get

$$\kappa_1 \|(\lambda - \mathbb{A}_{\tau,j})[\Pi_j^p f]\|_{1+\alpha} \geq |\lambda| \cdot \|\Pi_j^p f\|_{1+\alpha} + \|\Pi_j^p f\|_{2+\alpha} \quad (5.35)$$

for all $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geq \omega_1$, $\tau \in [0, 1]$, $f \in h^{2+\alpha}(\mathbb{S})$, and $j \in \{1, \dots, 2^{p+1}\}$. Combining (5.34) and (5.35) gives

$$\begin{aligned} \kappa_1 \|\Pi_j^p (\lambda - \partial_f \Phi_{1,\tau}^{\pi*})[f]\|_{1+\alpha} &\geq \kappa_1 \|\lambda \Pi_j^p f - \mathbb{A}_{\tau,j}[\Pi_j^p f]\|_{1+\alpha} - \kappa_1 \|\mathbb{A}_{\tau,j}[\Pi_j^p f] - \Pi_j^p \partial_f \Phi_{1,\tau}^{\pi*}[f]\|_{1+\alpha} \\ &\geq |\lambda| \cdot \|\Pi_j^p f\|_{1+\alpha} + \|\Pi_j^p f\|_{2+\alpha} - \frac{1}{2} \|\Pi_j^p f\|_{2+\alpha} - \kappa_1 K_2 \|f\|_{2+\alpha'}, \end{aligned}$$

hence

$$\kappa_1 K_2 \|f\|_{2+\alpha'} + \kappa_1 \|\Pi_j^p (\lambda - \partial_f \Phi_{1,\tau}^{\pi*})[f]\|_{1+\alpha} \geq |\lambda| \cdot \|\Pi_j^p f\|_{1+\alpha} + \frac{1}{2} \|\Pi_j^p f\|_{2+\alpha}$$

for all $\operatorname{Re} \lambda \geq \omega_1$, $f \in h^{2+\alpha}(\mathbb{S})$, $\tau \in [0, 1]$, and $j \in \{1, \dots, 2^{p+1}\}$. Together with Remark 5.4, we conclude that

$$\kappa_1 K_2 \|f\|_{2+\alpha'} + \kappa_1 \|(\lambda - \partial_f \Phi_{1,\tau}^{\pi*})[f]\|_{1+\alpha} \geq |\lambda| \cdot \|f\|_{1+\alpha} + \frac{1}{2} \|f\|_{2+\alpha}$$

for all $\operatorname{Re} \lambda \geq \omega_1$, $f \in h^{2+\alpha}(\mathbb{S})$, and $\tau \in [0, 1]$. Using the interpolation property (4.9) and Young's inequality, we find a constant $\tilde{\omega}_1 > 0$ depending on K_2 such that

$$4\kappa_1 \|(\lambda - \partial_f \Phi_{1,\tau}^{\pi*})[f]\|_{1+\alpha} \geq |\lambda| \cdot \|f\|_{1+\alpha} + \|f\|_{2+\alpha} \quad (5.36)$$

for all $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geq \tilde{\omega}_1$, $f \in h^{2+\alpha}(\mathbb{S})$, and $\tau \in [0, 1]$. Additionally, choosing σ sufficiently small we obtain from the definition of S_σ that

$$\tilde{\kappa}_1 := \max \left\{ 4\kappa_1, \sup \left\{ \|\partial_f \Phi_{1,1}^{\pi*}\|_{\mathcal{L}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}))} : (f_*, h_*) \in S_\sigma \cap \mathbb{B}_{(h^{2+\alpha}(\mathbb{S}))^2}((f_0, h_0), \sigma) \right\} \right\} < \infty.$$

Setting $\tau = 1$, we conclude that if $(\tilde{\omega}_1 - \partial_f \Phi_{1,1}^{\pi*}) = (\tilde{\omega}_1 - \partial_f \Phi_1^\pi(f_*, h_*)) \in \text{Isom}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}))$, then

$$-\partial_f \Phi_1^\pi(f_*, h_*) \in \mathcal{H}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}), \tilde{\kappa}_1, \tilde{\omega}_1),$$

where $\tilde{\kappa}_1$ is independent of $(f_*, h_*) \in (h^{3+\alpha}(\mathbb{S}))^2 \cap S_\sigma \cap \mathbb{B}_{(h^{2+\alpha}(\mathbb{S}))^2}((f_0, h_0), \sigma)$.

Hence, the assumptions of Lemma 5.2 hold true for sufficiently small σ if we can show that $(\tilde{\omega}_1 - \partial_f \Phi_{1,1}^{\pi*}) \in \text{Isom}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}))$. Using the method of continuity, cf. e.g. [26, Theorem 5.2], and (5.36) it follows that $(\lambda - \partial_f \Phi_{1,1}^{\pi*})$ is onto for $\text{Re } \lambda \geq \tilde{\omega}_1$, provided that $(\lambda - \partial_f \Phi_{1,0}^{\pi*})$ is onto. But the surjectivity of $(\lambda - \partial_f \Phi_{1,0}^{\pi*})$ for positive λ follows as in the proof of Proposition 5.8 below, and we thus have verified the assumptions of Lemma 5.2. \square

We present now a generation result for the Dirichlet-Neumann operator $\partial_f \Phi_{1,0}^{\pi*}$ in a slightly more general context.

Proposition 5.8. *Let $(f_*, h_*) \in \mathcal{V}$ be given and $a \in h^{2+\alpha}(\mathbb{S})$ be a positive function. The linear operator*

$$\mathbb{A}[f] := \mathcal{B}(f_*)w_{-0}^\pi[f], \quad f \in h^{2+\alpha}(\mathbb{S}),$$

with $(w_{+0}^\pi[f], w_{-0}^\pi[f])$ denoting the solution to

$$\left\{ \begin{array}{ll} \mathcal{A}_0^\pi(f_*, h_*)w_{+0}^\pi[f] = 0 & \text{in } \Omega_+, \\ \mathcal{A}_0^\pi(f_*)w_{-0}^\pi[f] = 0 & \text{in } \Omega_-, \\ \mathcal{B}(f_*, h_*)w_{+0}^\pi[f] - \mathcal{B}(f_*)w_{-0}^\pi[f] = 0 & \text{on } \Gamma_0, \\ w_{+0}^\pi[f] - w_{-0}^\pi[f] = af & \text{on } \Gamma_0, \\ w_{+0}^\pi[f] = 0 & \text{on } \Gamma_1, \\ w_{-0}^\pi[f] = 0 & \text{on } \Gamma_{-1}. \end{array} \right.$$

satisfies

$$-\mathbb{A} \in \mathcal{H}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})).$$

Proof. A short inspection of the proof of Theorem 5.5, Lemma 5.6, and Theorem 5.7 reveals that there exist constants $\kappa \geq 1$ and $\omega > 0$ such that

$$\kappa \|(\lambda - \mathbb{A})[f]\|_{1+\alpha} \geq |\lambda| \cdot \|f\|_{1+\alpha} + \|f\|_{2+\alpha}$$

for all $\lambda \in \mathbb{C}$ with $\text{Re } \lambda \geq \tilde{\omega}$ and all $f \in h^{2+\alpha}(\mathbb{S})$. To finish the proof, it suffices to prove that for all positive λ , the operator $(\lambda - \mathbb{A}) \in \mathcal{L}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}))$ is onto. Let thus $\lambda > 0$ and $F \in h^{1+\alpha}(\mathbb{S})$ be given, denote by $(z_+, z_-) \in h^{2+\alpha}(\Omega_+) \times h^{2+\alpha}(\Omega_-)$ the solution to the diffraction problem

$$\left\{ \begin{array}{ll} \mathcal{A}_0^\pi(f_*, h_*)z_+ = 0 & \text{in } \Omega_+, \\ \mathcal{A}_0^\pi(f_*)z_- = 0 & \text{in } \Omega_-, \\ \mathcal{B}(f_*, h_*)z_+ - \mathcal{B}(f_*)z_- = 0 & \text{on } \Gamma_0, \\ -\lambda a^{-1}(z_+ - z_-) + \mathcal{B}(f_*)z_- = F & \text{on } \Gamma_0, \\ z_+ = 0 & \text{on } \Gamma_1, \\ z_- = 0 & \text{on } \Gamma_{-1}, \end{array} \right. \quad (5.37)$$

and set $f := -a^{-1} \text{tr}_0(z_+ - z_-) \in h^{2+\alpha}(\mathbb{S})$. It is easy to see that $(w_{+0}^\pi, w_{-0}^\pi)[f] = (z_+, z_-)$. Therefore,

$$(\lambda - \mathbb{A})[f] = \lambda f + \mathcal{B}(f_*)w_{-0}^\pi[f] = -\lambda a^{-1}(z_+ - z_-) + \mathcal{B}(f_*)z_- = F.$$

In the remaining part of the proof, we establish that for each $F \in h^{1+\alpha}(\mathbb{S})$ and $\lambda > 0$, the diffraction problem (5.37) possesses a unique solution $(z_+, z_-) \in h^{2+\alpha}(\Omega_+) \times h^{2+\alpha}(\Omega_-)$. In fact, due to the arguments in the proof of Theorem 3.2, it suffices to prove that the mapping

$$(z_+, z_-) \mapsto \begin{pmatrix} \mathcal{A}_0^\pi(f_*, h_*)z_+ \\ \mathcal{A}_0^\pi(f_*)z_- \\ \mathcal{B}(f_*, h_*)z_+ - \mathcal{B}(f_*)z_- \\ -\lambda a^{-1} \text{tr}_0(z_+ - z_-) + \mathcal{B}(f_*)z_- \\ \text{tr}_1 z_+ \\ \text{tr}_{-1} z_- \end{pmatrix} \quad (5.38)$$

is an isomorphism between $C^{2+\alpha}(\overline{\Omega}_+) \times C^{2+\alpha}(\overline{\Omega}_-)$ and $C^\alpha(\overline{\Omega}_+) \times C^\alpha(\overline{\Omega}_-) \times (C^{1+\alpha}(\mathbb{S}))^2 \times (C^{2+\alpha}(\mathbb{S}))^2$. It is easily seen that for $\lambda = 0$ the operator (5.38) is an isomorphism between these spaces (as the third and fourth operators defined by (5.38) lead to decoupled equations). As the mapping

$$(z_+, z_-) \mapsto (0, 0, 0, -\lambda a^{-1} \text{tr}_0(z_+ - z_-), 0, 0)$$

is compact, we conclude that (5.38) defines a Fredholm operator of index zero. We are left to show that (5.38) defines for each $\lambda > 0$ an operator which is one-to-one. So, let (z_+, z_-) be a solution to (5.37) corresponding to $F = 0$ and assume that

$$\max_{\overline{\Omega}_-} z_- = z_-(x_0, 0) > 0.$$

If $z_- \not\equiv 0$, Hopf's lemma ensures that

$$(\mathcal{B}(f_*)z_-)(x_0) = \lambda(\Delta_\rho)^{-1}[z_+(x_0, 0) - z_-(x_0, 0)] > 0,$$

hence $\max_{\overline{\Omega}_+} z_+ > \max_{\overline{\Omega}_-} z_-$. On the other hand, if $\max_{\overline{\Omega}_+} z_+ = z_+(x_1, 0)$, then Hopf's lemma implies $(\mathcal{B}(f_*, h_*)z_+)(x_1) < 0$, whence $(z_+ - z_-)(x_1, 0) < 0$. This contradicts the inequality $\max_{\overline{\Omega}_+} z_+ > \max_{\overline{\Omega}_-} z_-$, meaning that $z_- \equiv 0$. But then also $z_+ \equiv 0$, and the proof is complete. \square

6. THE SECOND DIAGONAL OPERATOR

In this section we prove that the Fréchet derivative $\partial_h \Phi_2(0, (f_0, h_0))$ is the generator of a strongly continuous and analytic semigroup as stated in (3.17) when (f_0, h_0) and b_0 are such that the second inequality of (2.7) is satisfied. To derive the corresponding Theorem 6.5 we proceed in a similar way as in Section 5 and first identify the “leading order part” $\partial_h \Phi_2^\pi(f_0, h_0)$ of $\partial_h \Phi_2(0, (f_0, h_0))$. In Lemma 6.1 it is shown, however, that the latter is related to the solution operator of a Dirichlet problem which differs from the case considered in Section 5 where the leading order part $\partial_f \Phi_1^\pi$ was related to a diffraction problem. The arguments that follow are thus somewhat simpler than those in Section 5.

Lemma 6.1. *Let $(f_*, h_*) \in \mathcal{V}$ and let $\partial_h \Phi_2^\pi(f_*, h_*) \in \mathcal{L}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}))$ denote the operator defined by*

$$\partial_h \Phi_2^\pi(f_*, h_*)[h] := -\frac{k}{\mu_+} \left(\frac{2h'_* \text{tr}_1 \partial_y v_+^*}{h_* - f_*} - \text{tr}_1 \partial_x v_+^* \right) h' - \mathcal{B}_1(f_*, h_*) W_+^\pi[h], \quad h \in h^{2+\alpha}(\mathbb{S}), \quad (6.1)$$

where $(v_+^*, v_-^*) := (v_+^*, v_-^*)(f_*, h_*, b_0)$ is defined in (3.15) and where $W_+^\pi[h]$ denotes the solution to the Dirichlet problem

$$\begin{cases} \mathcal{A}_1^\pi(f_*, h_*)W_+^\pi[h] = \frac{\text{tr}_1 \partial_y v_+^*}{h_* - f_*} h'' & \text{in } \Omega_+, \\ W_+^\pi[h] = 0 & \text{on } \Gamma_0, \\ W_+^\pi[h] = g\rho_+ h & \text{on } \Gamma_1, \end{cases} \quad (6.2)$$

with

$$\mathcal{A}_1^\pi(f_*, h_*) := \partial_{xx} - \frac{2h_*'}{h_* - f_*} \partial_{xy} + \frac{h_*'^2 + 1}{(h_* - f_*)^2} \partial_{yy}. \quad (6.3)$$

Then, given $\varepsilon \in (0, 1)$, there exists $K_3 = K_3(\varepsilon) > 0$ such that

$$\|\partial_h \Phi_2(0, (f_*, h_*))[h] - \partial_h \Phi_2^\pi(f_*, h_*)[h]\|_{1+\alpha} \leq \varepsilon \|h\|_{2+\alpha} + K_3 \|h\|_{1+\alpha} \quad \text{for all } h \in h^{2+\alpha}(\mathbb{S}). \quad (6.4)$$

Moreover, $\partial_h \Phi_2^\pi \in C^\omega(\mathcal{V}, \mathcal{L}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})))$.

Proof. The regularity assertion follows by using Theorem 3.2. As for (6.4), let $\varepsilon \in (0, 1)$ be given and $\alpha' \in (0, \alpha)$ be fixed. Given $\delta \in (0, 1)$, we pick a cut-off function $\chi := \chi_\delta \in C^\infty([-1, 1])$ such that $0 \leq \chi \leq 1$, $\chi = 0$ for $y \leq 1 - \delta$, and $\chi = 1$ for $y \geq 1 - \delta/2$. We then obtain from (6.1), (3.12), and the Appendix that

$$\begin{aligned} \|\partial_h \Phi_2(0, (f_*, h_*))[h] - \partial_h \Phi_2^\pi(f_*, h_*)[h]\|_{1+\alpha} &\leq C(\|h\|_{1+\alpha} + \|\mathcal{B}_1(f_*, h_*)(W_+[h] - W_+^\pi[h])\|_{1+\alpha}) \\ &\leq C(\|h\|_{1+\alpha} + \|\text{tr}_1 \partial_y (W_+[h] - W_+^\pi[h])\|_{1+\alpha}) \\ &\leq C(\|h\|_{1+\alpha} + \|\chi(W_+[h] - W_+^\pi[h])\|_{2+\alpha}^{\Omega_+}), \end{aligned}$$

where $(W_+[h], W_-[h]) = (\partial_h v_+^*(f_*, h_*, b_0)[h], \partial_h v_-^*(f_*, h_*, b_0)[h])$ is the solution to (3.14) and C is independent of δ . We now notice that the pair $(u_+, u_-) := (W_+^\pi[h], 0) - (W_+[h], W_-[h])$ solves according to (6.2), (3.14), and the formulas for $\mathcal{A}(f_*, h_*)$ and $\partial_h \mathcal{A}(f_*, h_*)$ from the Appendix a diffraction problem of the form

$$\begin{cases} \mathcal{A}_1^\pi(f_*, h_*)u_+ = a_0^+ h + a_1^+ h' + (y \partial_y v_+^* - \text{tr}_1 \partial_y v_+^*) a_2^+ h'' \\ \quad + (y - 1)(b_0^+ \partial_{xy} W_+[h] + b_1^+ \partial_{yy} W_+[h]) + b_2^+ \partial_y W_+[h] & \text{in } \Omega_+, \\ \mathcal{A}(f_*)u_- = 0 & \text{in } \Omega_-, \\ \mathcal{B}(f_*, h_*)u_+ - \mathcal{B}(f_*)u_- = \partial_h \mathcal{B}(f_*, h_*)[h]v_+^* + \mathcal{B}(f_*, h_*)W_+^\pi[h] & \text{on } \Gamma_0, \\ u_+ - u_- = 0 & \text{on } \Gamma_0, \\ u_+ = 0 & \text{on } \Gamma_1, \\ u_- = 0 & \text{on } \Gamma_{-1}, \end{cases}$$

with $a_i^+, b_i^+ \in h^\alpha(\Omega_+)$, $0 \leq i \leq 2$. Therefore, $(u_+^\chi, u_-^\chi) := \chi(u_+, u_-)$ solves

$$\begin{cases} \mathcal{A}_1^\pi(f_*, h_*)u_+^\chi = \chi \mathcal{A}_1^\pi(f_*, h_*)u_+ - \frac{2h_*'}{h_* - f_*} \chi' \partial_x u_+ + \frac{h_*'^2 + 1}{(h_* - f_*)^2} (2\chi' \partial_y u_+ + \chi'' u_+) & \text{in } \Omega_+, \\ \mathcal{A}(f_*)u_-^\chi = 0 & \text{in } \Omega_-, \\ \mathcal{B}(f_*, h_*)u_+^\chi - \mathcal{B}(f_*)u_-^\chi = 0 & \text{on } \Gamma_0, \\ u_+^\chi - u_-^\chi = 0 & \text{on } \Gamma_0, \\ u_+^\chi = 0 & \text{on } \Gamma_1, \\ u_-^\chi = 0 & \text{on } \Gamma_{-1}. \end{cases}$$

Recalling the Schauder estimate (3.6), we obtain that

$$\begin{aligned} \|u_+^\chi\|_{2+\alpha}^{\Omega_+} &\leq C\|\chi\mathcal{A}_1^\pi(f_*, h_*)u_+\|_\alpha^{\Omega_+} + C(\delta)\|u_+\|_{2+\alpha'}^{\Omega_+} \\ &\leq C(\|\chi(y-1)(b_0^+\partial_{xy}W_+[h] + b_1^+\partial_{yy}W_+[h])\|_\alpha^{\Omega_+} + \|\chi(y\partial_y v_+^* - \text{tr}_1 \partial_y v_+^*)a_2^+ h''\|_\alpha^{\Omega_+}) \\ &\quad + C(\delta)\|h\|_{2+\alpha'} \\ &\leq C(\|\chi(y-1)\|_0^{\Omega_+} + \|\chi(y\partial_y v_+^* - \text{tr}_1 \partial_y v_+^*)\|_0^{\Omega_+})\|h\|_{2+\alpha} + C(\delta)\|h\|_{2+\alpha'}, \end{aligned}$$

and choosing $\delta > 0$ such that $\|\chi(y-1)\|_0^{\Omega_+} + \|\chi(y\partial_y v_+^* - \text{tr}_1 \partial_y v_+^*)\|_0^{\Omega_+} < \varepsilon/2C$, interpolation properties of small Hölder spaces lead us to the desired estimate (6.4). \square

As in the previous section we introduce for $\sigma > 0$ the set R_σ consisting of those $(f_*, h_*) \in \mathcal{V}$ satisfying the inequalities

$$\begin{aligned} (1) \quad &\sigma < \min\{f_* - d, h_* - f_*\}, \quad \|f_*\|_2 + \|h_*\|_2 < \sigma^{-1}, \\ (2) \quad &g\rho_+ > \sigma + \frac{\text{tr}_1 \partial_y v_+^*}{h_* - f_*}, \\ (3) \quad &\|\text{tr}_1 \partial_y v_+^*\|_0 < \sigma^{-1}, \end{aligned}$$

where (v_+^*, v_-^*) is defined in (3.15). Since the functions (f_0, h_0) and b_0 are chosen such that the second inequality in (2.7) is satisfied, we may choose σ such that $(f_0, h_0) \in R_\sigma$.

Lemma 6.2. *Let $\sigma > 0$ be such that $(f_0, h_0) \in R_\sigma$. Assume that there exists a constant $\tilde{\kappa}_2 := \tilde{\kappa}_2(\sigma)$ and for each $(f_*, h_*) \in (h^{3+\alpha}(\mathbb{S}))^2 \cap R_\sigma \cap \mathbb{B}_{(h^{2+\alpha}(\mathbb{S}))^2}((f_0, h_0), \sigma)$ there exists a further constant $\tilde{\omega}_2 > 0$ with the property that*

$$-\partial_h \Phi_2^\pi(f_*, h_*) \in \mathcal{H}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}), \tilde{\kappa}_2, \tilde{\omega}_2).$$

It then holds

$$-\partial_h \Phi_2^\pi(f_0, h_0) \in \mathcal{H}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})).$$

Proof. The proof is similar to that of Lemma 5.2. \square

We are thus left to prove that there exists $\sigma > 0$ such that $\partial_h \Phi_2^\pi(f_*, h_*)$ is an analytic generator for each $(f_*, h_*) \in (h^{3+\alpha}(\mathbb{S}))^2 \cap R_\sigma \cap \mathbb{B}_{(h^{2+\alpha}(\mathbb{S}))^2}((f_0, h_0), \sigma)$, with the constant $\tilde{\kappa}_2$ depending only on σ . To this end we use the additional regularity of $(f_*, h_*) \in (h^{3+\alpha}(\mathbb{S}))^2 \cap R_\sigma$ to introduce again a parameter $\tau \in [0, 1]$ which enables us to continuously transform the leading order part $\partial_h \Phi_2^{\pi*} := \partial_h \Phi_2^\pi(f_*, h_*)$ into a (negative) Dirichlet-Neumann map. More precisely, for each $\tau \in [0, 1]$ we define the operator $\partial_h \Phi_{2,\tau}^\pi \in \mathcal{L}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}))$ by the formula

$$\partial_h \Phi_{2,\tau}^{\pi*}[h] := -\frac{\tau k}{\mu_+} \left(\frac{2h_*' \text{tr}_1 \partial_y v_+^*}{h_* - f_*} - \text{tr}_1 \partial_x v_+^* \right) h' - \mathcal{B}_1(f_*, h_*) W_{+\tau}^\pi[h], \quad h \in h^{2+\alpha}(\mathbb{S}), \quad (6.5)$$

with $W_{+\tau}^\pi[h]$ denoting the solution to

$$\begin{cases} \mathcal{A}_1^\pi(f_*, h_*) W_{+\tau}^\pi[h] = \frac{\tau \text{tr}_1 \partial_y v_+^*}{h_* - f_*} h'' & \text{in } \Omega_+, \\ W_{+\tau}^\pi[h] = 0 & \text{on } \Gamma_0, \\ W_{+\tau}^\pi[h] = \left[g\rho_+ - (1-\tau) \frac{\text{tr}_1 \partial_y v_+^*}{h_* - f_*} \right] h & \text{on } \Gamma_1. \end{cases} \quad (6.6)$$

For $\tau = 1$ we see that $\partial_h \Phi_{2,1}^{\pi*} = \partial_h \Phi_2^\pi(f_*, h_*)$, while for $\tau = 0$ we obtain a Dirichlet-Neumann operator.

We now prove the following perturbation result.

Theorem 6.3. *Let $\sigma > 0$ be such that $(f_0, h_0) \in R_\sigma$ and let $\mu > 0$ and $\alpha' \in (0, \alpha)$ be given. Then, given $(f_*, h_*) \in (h^{3+\alpha}(\mathbb{S}))^2 \cap R_\sigma$, there exist an integer $p \geq 3$, a p -partition of unity $\{\Pi_j^p\}_{1 \leq j \leq 2^{p+1}}$, and a constant $K_4 = K_4(p)$, and for each $\tau \in [0, 1]$ and $1 \leq j \leq 2^{p+1}$ there are bounded operators $\mathbb{B}_{\tau,j} \in \mathcal{L}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}))$ such that*

$$\|\Pi_j^p \partial_h \Phi_{2,\tau}^{\pi*}[h] - \mathbb{B}_{\tau,j}[\Pi_j^p h]\|_{1+\alpha} \leq \mu \|\Pi_j^p h\|_{2+\alpha} + K_4 \|h\|_{2+\alpha'} \quad (6.7)$$

for all $h \in h^{2+\alpha}(\mathbb{S})$. The operators $\mathbb{B}_{\tau,j}$ are defined by the formula

$$\mathbb{B}_{\tau,j}[h] := -\frac{\tau k}{\mu_+} \left(\frac{2h'_* \operatorname{tr}_1 \partial_y v_+^*}{h_* - f_*} - \operatorname{tr}_1 \partial_x v_+^* \right) \Big|_{x_j^p} h' - \frac{k}{\mu_+} \left(\frac{1 + h_*'^2}{h_* - f_*} \Big|_{x_j^p} \operatorname{tr}_1 \partial_y W_{+\tau}^{\pi,j}[h] - h'_*(x_j^p) \operatorname{tr}_1 \partial_x W_{+\tau}^{\pi,j}[h] \right), \quad (6.8)$$

where $W_{+\tau}^{\pi,j}[h]$ denotes the solution to the problem

$$\begin{cases} \mathcal{A}_{1,j}^\pi(f_*, h_*) W_{+\tau}^{\pi,j}[h] = \frac{\tau \operatorname{tr}_1 \partial_y v_+^*}{h_* - f_*} \Big|_{x_j^p} h'' & \text{in } \Omega_+, \\ W_{+\tau}^{\pi,j}[h] = 0 & \text{on } \Gamma_0, \\ W_{+\tau}^{\pi,j}[h] = \left[g\rho_+ - (1 - \tau) \frac{\operatorname{tr}_1 \partial_y v_+^*}{h_* - f_*} \Big|_{x_j^p} \right] h & \text{on } \Gamma_1, \end{cases} \quad (6.9)$$

with $\mathcal{A}_{1,j}^\pi(f_*, h_*)$ being the operator obtained from $\mathcal{A}_1^\pi(f_*, h_*)$ when evaluating its coefficients at x_j^p .

Proof. Let $\mu > 0$ and $\alpha' \in (0, \alpha)$ be fixed. Given $p \geq 3$, a p -partition of unity $\{\Pi_j^p\}_{1 \leq j \leq 2^{p+1}}$, and an associated family $\{\chi_j^p\}_{1 \leq j \leq 2^{p+1}}$ (see Section 5) we decompose

$$\Pi_j^p \partial_h \Phi_{2,\tau}^{\pi*}[h] - \mathbb{B}_{\tau,j}[\Pi_j^p h] = S_1 + S_2 + S_3$$

with

$$\begin{aligned} S_1 &:= \frac{(1 + \tau)k g\rho_+}{\mu_+} (\Pi_j^p h'_* h' - h'_*(x_j^p) (\Pi_j^p h)'), \\ S_2 &:= \frac{(1 + \tau)k}{\mu_+} \left[\frac{h'_* \operatorname{tr}_1 \partial_y v_+^*}{h_* - f_*} \Big|_{x_j^p} (\Pi_j^p h)' - \frac{h'_* \operatorname{tr}_1 \partial_y v_+^*}{h_* - f_*} \Pi_j^p h' \right], \\ S_3 &:= \frac{k}{\mu_+} \frac{1 + h_*'^2}{h_* - f_*} \Big|_{x_j^p} \operatorname{tr}_1 \partial_y W_{+\tau}^{\pi,j}[\Pi_j^p h] - \frac{k}{\mu_+} \frac{1 + h_*'^2}{h_* - f_*} \Pi_j^p \operatorname{tr}_1 \partial_y W_{+\tau}^\pi[h]. \end{aligned}$$

Still, we use C for constants which are independent of p , constants depending on p being denoted by K . Recalling that $\chi_j^p \Pi_j^p = \Pi_j^p$, we estimate S_1 as

$$\begin{aligned} \|S_1\|_{1+\alpha} &\leq C \|\Pi_j^p h'_* h' - h'_*(x_j^p) (\Pi_j^p h)'\|_{1+\alpha} \leq C \|\Pi_j^p h'_* \chi_j^p (h'_* - h'_*(x_j^p))\|_{1+\alpha} + K \|h\|_{1+\alpha} \\ &\leq C \|\chi_j^p (h'_* - h'_*(x_j^p))\|_0 \|\Pi_j^p h\|_{2+\alpha} + K \|h\|_2, \end{aligned}$$

and similarly

$$\|S_2\|_{1+\alpha} \leq C \left\| \chi_j^p \left(\frac{2h'_* \operatorname{tr}_1 \partial_y v_+^*}{h_* - f_*} \Big|_{x_j^p} - \frac{2h'_* \operatorname{tr}_1 \partial_y v_+^*}{h_* - f_*} \right) \right\|_0 \|\Pi_j^p h\|_{2+\alpha} + K \|h\|_2.$$

Choosing p sufficiently large, we find that

$$\|S_1\|_{1+\alpha} + \|S_2\|_{1+\alpha} \leq \frac{\mu}{2} \|\Pi_j^p h\|_{2+\alpha} + K \|h\|_{1+\alpha}. \quad (6.10)$$

We are left to estimate S_3 . For this we note that

$$\begin{aligned} \|S_3\|_{1+\alpha} &\leq C \left\| \frac{1+h_*'^2}{h_*-f_*} \Big|_{x_j^p} \operatorname{tr}_1 \partial_y W_{+\tau}^{\pi,j} [\Pi_j^p h] - \frac{1+h_*'^2}{h_*-f_*} \Pi_j^p \operatorname{tr}_1 \partial_y W_{+\tau}^\pi [h] \right\|_{1+\alpha} \\ &\leq C \left\| (1-\chi_j^p) \frac{1+h_*'^2}{h_*-f_*} \Big|_{x_j^p} \operatorname{tr}_1 \partial_y W_{+\tau}^{\pi,j} [\Pi_j^p h] \right\|_{1+\alpha} \\ &\quad + \left\| \chi_j^p \frac{1+h_*'^2}{h_*-f_*} \Big|_{x_j^p} \operatorname{tr}_1 \partial_y W_{+\tau}^{\pi,j} [\Pi_j^p h] - \chi_j^p \frac{1+h_*'^2}{h_*-f_*} \operatorname{tr}_1 \partial_y (\Pi_j^p W_{+\tau}^\pi [h]) \right\|_{1+\alpha} =: S_{31} + S_{32}, \end{aligned}$$

where

$$\|S_{31}\|_{1+\alpha} \leq C \left\| \operatorname{tr}_1 \partial_y ((1-\chi_j^p) W_{+\tau}^{\pi,j} [\Pi_j^p h]) \right\|_{1+\alpha} \leq C \|(1-\chi_j^p) W_{+\tau}^{\pi,j} [\Pi_j^p h]\|_{2+\alpha}^{\Omega+}.$$

As $z_+ := (1-\chi_j^p) W_{+\tau}^{\pi,j} [\Pi_j^p h]$ solves according to (6.9) and (6.3) the problem

$$\left\{ \begin{array}{ll} \mathcal{A}_{1,j}^\pi(f_*, h_*) z_+ = \frac{\tau \operatorname{tr}_1 \partial_y v_+^*}{h_*-f_*} \Big|_{x_j^p} (1-\chi_j^p) (\Pi_j^p h)'' - (\chi_j^p)'' W_{+\tau}^{\pi,j} [\Pi_j^p h] \\ \quad - 2(\chi_j^p)' \partial_x W_{+\tau}^{\pi,j} [\Pi_j^p h] + \frac{2h_*'}{h_*-f_*} \Big|_{x_j^p} (\chi_j^p)' \partial_y W_{+\tau}^{\pi,j} [\Pi_j^p h] & \text{in } \Omega_+, \\ z_+ = 0 & \text{on } \Gamma_0, \\ z_+ = \left[g\rho_+ - (1-\tau) \frac{\operatorname{tr}_1 \partial_y v_+^*}{h_*-f_*} \Big|_{x_j^p} \right] (1-\chi_j^p) \Pi_j^p h & \text{on } \Gamma_1, \end{array} \right.$$

we infer from $(1-\chi_j^p) \Pi_j^p = 0$ and Schauder estimates for elliptic Dirichlet problems that

$$\|S_{31}\|_{1+\alpha} \leq K \|h\|_{2+\alpha}. \quad (6.11)$$

Finally,

$$\|S_{32}\|_{1+\alpha} \leq C \left\| \chi_j^p \left(\frac{1+h_*'^2}{h_*-f_*} \Big|_{x_j^p} - \frac{1+h_*'^2}{h_*-f_*} \right) \right\|_0 \|\Pi_j^p h\|_{2+\alpha} + C \|\chi_j^p (W_{+\tau}^{\pi,j} [\Pi_j^p h] - \Pi_j^p W_{+\tau}^\pi [h])\|_{2+\alpha}^{\Omega+},$$

and for p large enough we have

$$\left\| \chi_j^p \left(\frac{1+h_*'^2}{h_*-f_*} \Big|_{x_j^p} - \frac{1+h_*'^2}{h_*-f_*} \right) \right\|_0 \leq \frac{\mu}{4C}. \quad (6.12)$$

On the other hand, $u_+ := \chi_j^p(W_{+\tau}^{\pi,j}[\Pi_j^p h] - \Pi_j^p W_{+\tau}^\pi[h])$ solves the Dirichlet problem

$$\left\{ \begin{array}{ll} \mathcal{A}_1^\pi(f_*, h_*)u_+ = (\mathcal{A}_1^\pi(f_*, h_*) - \mathcal{A}_{1,j}^\pi(f_*, h_*))[\chi_j^p W_{+\tau}^{\pi,j}[\Pi_j^p h]] \\ \quad + \frac{\tau \operatorname{tr}_1 \partial_y v_+^*}{h_* - f_*} \Big|_{x_j^p} \chi_j^p (\Pi_j^p h)'' - \frac{\tau \operatorname{tr}_1 \partial_y v_+^*}{h_* - f_*} \Pi_j^p h'' \\ \quad + (\chi_j^p)'' W_{+\tau}^{\pi,j}[\Pi_j^p h] - (\Pi_j^p)'' W_{+\tau}^\pi[h] \\ \quad + 2(\chi_j^p)' \partial_x W_{+\tau}^{\pi,j}[\Pi_j^p h] - 2(\Pi_j^p)' \partial_x W_{+\tau}^\pi[h] \\ \quad - \frac{2h_*'}{h_* - f_*} \Big|_{x_j^p} (\chi_j^p)' \partial_y W_{+\tau}^{\pi,j}[\Pi_j^p h] + \frac{2h_*'}{h_* - f_*} (\Pi_j^p)' \partial_y W_{+\tau}^\pi[h] \quad \text{in } \Omega_+, \\ u_+ = (1 - \tau) \chi_j^p \left(\frac{\operatorname{tr}_1 \partial_y v_+^*}{h_* - f_*} - \frac{\operatorname{tr}_1 \partial_y v_+^*}{h_* - f_*} \Big|_{x_j^p} \right) \Pi_j^p h \quad \text{on } \Gamma_0, \\ u_+ = 0 \quad \text{on } \Gamma_1, \end{array} \right.$$

and therefore

$$\begin{aligned} \|u_+\|_{2+\alpha}^{\Omega_+} &\leq C \left(\|(\mathcal{A}_1^\pi(f_*, h_*) - \mathcal{A}_{1,j}^\pi(f_*, h_*))[\chi_j^p W_{+\tau}^{\pi,j}[\Pi_j^p h]]\|_\alpha \right. \\ &\quad + \left\| \chi_j^p \left(\frac{\operatorname{tr}_1 \partial_y v_+^*}{h_* - f_*} - \frac{\operatorname{tr}_1 \partial_y v_+^*}{h_* - f_*} \Big|_{x_j^p} \right) \right\|_0 \|\Pi_j^p h''\|_\alpha \\ &\quad + \left\| \chi_j^p \left(\frac{\operatorname{tr}_1 \partial_y v_+^*}{h_* - f_*} - \frac{\operatorname{tr}_1 \partial_y v_+^*}{h_* - f_*} \Big|_{x_j^p} \right) \right\|_0 \|\Pi_j^p h\|_{2+\alpha} \\ &\quad \left. + K \|h\|_{2+\alpha'} \right). \end{aligned}$$

For p sufficiently large, we obtain together with (6.12) that

$$\|S_{32}\|_{1+\alpha} \leq \frac{\mu}{4} \|\Pi_j^p h\|_{2+\alpha} + C \|u_+\|_{2+\alpha}^{\Omega_+} \leq \frac{\mu}{2} \|\Pi_j^p h\|_{2+\alpha} + K \|h\|_{2+\alpha'}. \quad (6.13)$$

Gathering (6.10), (6.11), and (6.13), the desired estimate (6.7) follows. \square

Fourier analysis: The symbol of $\mathbb{B}_{\tau,j}$. Let $\tau \in [0, 1]$, $p, j \in \mathbb{N}$ with $p \geq 3$ and $1 \leq j \leq 2^{p+1}$ be arbitrary. Given $h \in h^{2+\alpha}(\mathbb{S})$, we consider its Fourier expansion

$$h = \sum_{m \in \mathbb{Z}} h_m e^{imx}$$

and look for the corresponding expansion of $\mathbb{B}_{\tau,j}[h]$. Recalling the definition of $\mathbb{B}_{\tau,j}$ from (6.8), we first determine the expansions for $\operatorname{tr}_1 \nabla W_{+\tau}^{\pi,j}[h]$. Let

$$W_{+\tau}^{\pi,j}[h] = \sum_{m \in \mathbb{Z}} B_m h_m e^{imx} \quad (6.14)$$

with functions $B_m = B_m(y)$ to be determined. Recalling (6.3), we let (using the summation convention)

$$\mathcal{A}_{1,j}^\pi(f_*, h_*) = c_{pl} \partial_{pl}$$

and set

$$a := c_{12}/c_{22}, \quad b := 1/c_{22}, \quad D := \sqrt{b - a^2}$$

noticing that $b - a^2 > 0$. Since $W_{+\tau}^{\pi,j}[h]$ is the solution to (6.9), it follows from (6.14) that for each $m \in \mathbb{Z}$, the function B_m solves the problem

$$\begin{cases} (B_m)'' + 2mia(B_m)' - bm^2 B_m = -\tau b V m^2 & \text{in } 0 < y < 1, \\ B_m(0) = 0, \\ B_m(1) = g\rho_+ - (1 - \tau)V, \end{cases} \quad (6.15)$$

where

$$V := \frac{\text{tr}_1 \partial_y v_+}{h_* - f_*} \Big|_{x_j^p}. \quad (6.16)$$

The general solution of the first equation of (6.15) is given by the formula

$$B_m := u_m + iv_m \quad (6.17)$$

with real part

$$\begin{aligned} u_m(y) = & \zeta_1 \left(\cos(amy) \cosh(Dmy) + \frac{a}{D} \sin(amy) \sin(Dmy) \right) + \frac{\zeta_2}{Dm} \cos(amy) \sin(Dmy) \\ & + \zeta_3 \left(\sin(amy) \cosh(Dmy) - \frac{a}{D} \cos(amy) \sin(Dmy) \right) + \frac{\zeta_4}{Dm} \sin(amy) \sin(Dmy) + \tau V \end{aligned}$$

and imaginary part

$$\begin{aligned} v_m(y) = & \zeta_1 \left(-\sin(amy) \cosh(Dmy) + \frac{a}{D} \cos(amy) \sin(Dmy) \right) - \frac{\zeta_2}{Dm} \sin(amy) \sin(Dmy) \\ & + \zeta_3 \left(\cos(amy) \cosh(Dmy) + \frac{a}{D} \sin(amy) \sin(Dmy) \right) + \frac{\zeta_4}{Dm} \cos(amy) \sin(Dmy). \end{aligned}$$

The real constants $\{\zeta_i : 1 \leq i \leq 4\}$ are to be determined such that the last two equations of (6.15) are also satisfied by B_m . Explicit computations (much simpler compared to those in Section 5) show that such constants can be uniquely determined to obtain B_m . Furthermore, using the expressions for ζ_i it follows again by explicit computations that the operator $\mathbb{B}_{\tau,j}$ from (6.8) is a Fourier multiplier with symbol $(\varphi_m)_m$, the real part being given by

$$\text{Re } \varphi_m := -\mu_m - \nu_m, \quad (6.18)$$

where

$$\mu_m := \frac{k}{\mu_+} \frac{g\rho_+ - V}{\tanh(Dm)/m} \quad \text{and} \quad \nu_m := \frac{\tau k V}{\mu_+} \frac{\cos(am)}{\sinh(Dm)/m}.$$

The imaginary part is given by

$$\text{Im } \varphi_m = \frac{\tau k}{\mu_+} \left[\frac{a((2 - \tau)V - g\rho_+)}{D} m + \frac{V \sin(am)}{\sinh(Dm)/m} \right]. \quad (6.19)$$

The representation of $\mathbb{B}_{\tau,j}$ allows us now to prove the following generation result.

Lemma 6.4. *Let $\sigma > 0$ be such that $(f_0, h_0) \in R_\sigma$. Then, there exist constants $\kappa_2 \geq 1$ and $\omega_2 > 0$, depending only σ such that for all $(f_*, h_*) \in (h^{3+\alpha}(\mathbb{S}))^2 \cap R_\sigma$, any p -partition of unity $\{\Pi_j^p\}_{1 \leq j \leq 2^{p+1}}$, $p \in \mathbb{N}$ with $p \geq 3$, and any $\tau \in [0, 1]$, the operators $\mathbb{B}_{\tau,j}$, $1 \leq j \leq 2^{p+1}$, defined by (6.8) satisfy*

$$\lambda - \mathbb{B}_{\tau,j} \in \text{Isom}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})), \quad (6.20)$$

$$\kappa_2 \|(\lambda - \mathbb{B}_{\tau,j})[h]\|_{1+\alpha} \geq |\lambda| \cdot \|h\|_{1+\alpha} + \|h\|_{2+\alpha} \quad (6.21)$$

for all $h \in h^{2+\alpha}(\mathbb{S})$ and $\lambda \in \mathbb{C}$ with $\text{Re } \lambda \geq \omega_2$.

Proof. Recalling (6.16) and the definition of R_σ , we find a constant $C_0 > 0$ depending on σ such that

$$C_0^{-1} \leq g\rho_+ - V \leq C_0,$$

and such that the relations (5.28) are satisfied when replacing (λ_m) by (φ_m) . With this observation, the desired result follows along the lines of the proof of Lemma 5.6. \square

We are now in a position to establish (3.17).

Theorem 6.5. *Let $(f_0, h_0) \in \mathcal{V}$ and $b_0 \in h^{2+\alpha}(\mathbb{S})$ be given such that the second inequality in (2.7) is satisfied. Then*

$$-\partial_h \Phi_2(0, (f_0, h_0)) \in \mathcal{H}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})).$$

Proof. Recalling (6.4) (with $(f_*, h_*) = (f_0, h_0)$) and [2, Theorem I.1.3.1 (ii)], we are left to show that $-\partial_h \Phi_2^\pi(f_0, h_0) \in \mathcal{H}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}))$. For this, we prove that the assumptions of Lemma 6.2 are satisfied provided that σ is sufficiently small.

Let $\sigma > 0$ be such that $(f_0, h_0) \in R_\sigma$, let κ_2 and ω_2 be the constants found in Lemma 6.4, and pick $\alpha' \in (0, \alpha)$. Given $(f_*, h_*) \in (h^{3+\alpha}(\mathbb{S}))^2 \cap R_\sigma$ and using the same arguments as in the proofs of Theorem 5.7, Theorem 6.3, and Lemma 6.4, it follows that there exists a constant $\tilde{\omega}_2 > 1$ such that

$$4\kappa_2 \|(\lambda - \partial_h \Phi_{2,\tau}^{\pi*})[h]\|_{1+\alpha} \geq |\lambda| \cdot \|h\|_{1+\alpha} + \|h\|_{2+\alpha} \quad (6.22)$$

for $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geq \tilde{\omega}_2$, $h \in h^{2+\alpha}(\mathbb{S})$, and $\tau \in [0, 1]$. As in the proof of Theorem 5.7 we may now choose σ small to find a constant $\tilde{\kappa}_2 \geq 1$ depending only on σ such that, for (f_*, h_*) belonging additionally to the ball $\mathbb{B}_{(h^{2+\alpha}(\mathbb{S}))^2}((f_0, h_0), \sigma)$, we have

$$-\partial_h \Phi_2^\pi(f_*, h_*) \in \mathcal{H}(h^{2+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}), \tilde{\kappa}_2, \tilde{\omega}_2),$$

provided $(\tilde{\omega}_2 - \partial_h \Phi_{2,1}^{\pi*}) = (\tilde{\omega}_2 - \partial_h \Phi_2^\pi(f_*, h_*))$ is an isomorphism.

We now establish this last property. Note that, due to (6.22), $(\tilde{\omega}_2 - \partial_h \Phi_{2,1}^{\pi*})$ is an isomorphism if $(\tilde{\omega}_2 - \partial_2 \Phi_{2,0}^{\pi*})$ is onto. To prove the latter let $\lambda > 0$ and $H \in h^{1+\alpha}(\mathbb{S})$ be given. We let $z \in h^{2+\alpha}(\Omega_+)$ be the unique solution to the elliptic boundary value problem

$$\begin{cases} \mathcal{A}_1^\pi(f_*, h_*)z = 0 & \text{in } \Omega_+, \\ z = 0 & \text{on } \Gamma_0, \\ \lambda \left(g\rho_+ - \frac{\operatorname{tr}_1 \partial_y v_+^*}{h_* - f_*} \right)^{-1} z + \mathcal{B}_1(f_*, h_*)z = H & \text{on } \Gamma_1. \end{cases} \quad (6.23)$$

We then set

$$h := \left(g\rho_+ - \frac{\operatorname{tr}_1 \partial_y v_+^*}{h_* - f_*} \right)^{-1} \operatorname{tr}_1 z \in h^{2+\alpha}(\mathbb{S}).$$

Recalling the definitions (6.5) and (6.6), it follows that $W_{+0}^\pi[h] = z$ and

$$(\lambda - \partial_h \Phi_{2,0}^{\pi*})[h] = \lambda h + \mathcal{B}_1(f_*, h_*)W_{+0}^\pi[h] = \lambda \left(g\rho_+ - \frac{\operatorname{tr}_1 \partial_y v_+^*}{h_* - f_*} \right)^{-1} \operatorname{tr}_1 z + \mathcal{B}_1(f_*, h_*)z = H.$$

In view of Lemma 6.2 the proof is complete. \square

Finally, we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Because of the equivalence of the problems (2.3) and (3.1) and the reduction of the latter to (3.8), we are left to investigate (3.8). For the existence and uniqueness result we shall employ an abstract result for fully nonlinear problems [30, Theorem 8.4.1]. Let us first note that (3.11) is implied by Lemma 4.1, Theorems 5.7, 6.5, and [2, Theorem I.1.6.1 and Remark I.1.6.2]. Thus, the assumptions of [30, Theorem 8.4.1] are satisfied owing to (3.10), (3.11), and the interpolation property (4.9). This proves the existence and uniqueness part.

Finally, the continuous dependence of the solution on the initial data follows from [30, Theorem 8.4.4]. \square

7. THE MUSKAT PROBLEM WITH SURFACE TENSION EFFECTS

In this section we investigate the Muskat problem introduced in Section 2 when allowing for surface tension effects in the presence or absence of gravity. More precisely, instead of being continuous along the interfaces $\Gamma(f)$ and $\Gamma(h)$, we assume that the pressure jump along an interface is proportional to the curvature of the respective interface, i.e., the pressure obeys the Laplace-Young equation

$$\begin{aligned} p_- - p_+ &= -\gamma_f \kappa(f) \quad \text{on } \Gamma(f), \\ p_+ &= -\gamma_h \kappa(h) \quad \text{on } \Gamma(h), \end{aligned} \tag{7.1}$$

where γ_f [resp. γ_h] is the surface tension coefficient at the interface $\Gamma(f)$ [resp. $\Gamma(h)$] and where for each $\zeta \in C^2(\mathbb{S})$ the function

$$\kappa(\zeta) := \frac{\zeta''}{(1 + \zeta'^2)^{3/2}}$$

is the curvature of the graph $[y = \zeta]$.

Hence, instead of (2.3a) we consider the system

$$\left\{ \begin{array}{ll} \Delta u_+ = 0 & \text{in } \Omega(f, h), \\ \Delta u_- = 0 & \text{in } \Omega(f), \\ \partial_t h = -k\mu_+^{-1} \sqrt{1 + h'^2} \partial_\nu u_+ & \text{on } \Gamma(h), \\ u_+ = g\rho_+ h - \gamma_h \kappa(h) & \text{on } \Gamma(h), \\ u_- = b & \text{on } \Gamma_d, \\ u_+ - u_- = g(\rho_+ - \rho_-)f + \gamma_f \kappa(f) & \text{on } \Gamma(f), \\ \partial_t f = -k\mu_\pm^{-1} \sqrt{1 + f'^2} \partial_\nu u_\pm & \text{on } \Gamma(f), \end{array} \right. \tag{7.2}$$

supplemented with the initial conditions (2.3b).

The main result regarding problem (7.2) is following theorem. Its proof is given at the end of this section.

Theorem 7.1. *Let $g \geq 0$, $(f_0, h_0) \in \mathcal{V} \cap (h^{4+\alpha}(\mathbb{S}))^2$, and b be given such that (2.4) holds. Then, there exist a maximal existence time $T_0 := T_0(f_0, h_0) \in (0, T]$ and a unique classical Hölder solution² (f, h, u_+, u_-) of (7.2) and (2.3b) on $[0, T_0)$. Additionally, the solutions depend continuously on the initial data.*

In order to prove Theorem 7.1, we first recast the problem as a nonlinear and nonlocal evolution equation. To this end, we infer from the proof of Theorem 3.2 that for each pair of functions $(f, h) \in \mathcal{V} \cap (h^{4+\alpha}(\mathbb{S}))^2$ and $b \in h^{2+\alpha}(\mathbb{S})$ there exists a unique solution

$$(v_+, v_-) := (v_+(f, h, b), v_-(f, h, b)) \in h^{2+\alpha}(\Omega_+) \times h^{2+\alpha}(\Omega_-)$$

²The notion of classical solution for (7.2) and (2.3b) is the same as in Section 2 with the modification that we require additionally to (2.5) that $(f, h) \in C([0, T_0), (h^{4+\alpha}(\mathbb{S}))^2)$.

of the diffraction problem

$$\left\{ \begin{array}{ll} \mathcal{A}(f, h)v_+ = 0 & \text{in } \Omega_+, \\ \mathcal{A}(f)v_- = 0 & \text{in } \Omega_-, \\ \mathcal{B}(f, h)v_+ - \mathcal{B}(f)v_- = 0 & \text{on } \Gamma_0, \\ v_+ - v_- = g(\rho_+ - \rho_-)f + \gamma_f \kappa(f) & \text{on } \Gamma_0, \\ v_+ = g\rho_+ h - \gamma_h \kappa(h) & \text{on } \Gamma_1, \\ v_- = b & \text{on } \Gamma_{-1}, \end{array} \right. \quad (7.3)$$

the mapping

$$[(f, h, b) \mapsto (v_+(f, h, b), v_-(f, h, b))] \in C^\omega((\mathcal{V} \cap (h^{4+\alpha}(\mathbb{S}))^2) \times h^{2+\alpha}(\mathbb{S}), h^{2+\alpha}(\Omega_+) \times h^{2+\alpha}(\Omega_-))$$

being real-analytic as a consequence of $\kappa \in C^\omega(h^{4+\alpha}(\mathbb{S}), h^{2+\alpha}(\mathbb{S}))$. Using this observation, we deduce that the problem (7.2) and (2.3b) is equivalent to the evolution equation

$$\partial_t(f, h) = \Phi(t, (f, h)), \quad (7.4)$$

where $\Phi : [0, T) \times (\mathcal{V} \cap (h^{4+\alpha}(\mathbb{S}))^2) \subset \mathbb{R} \times (h^{4+\alpha}(\mathbb{S}))^2 \rightarrow (h^{1+\alpha}(\mathbb{S}))^2$ is the operator $\Phi = (\Phi_1, \Phi_2)$ defined by

$$\begin{aligned} \Phi_1(t, (f, h)) &:= -\mathcal{B}(f)v_-(f, h, b(t)), \\ \Phi_2(t, (f, h)) &:= -\mathcal{B}_1(f, h)v_+(f, h, b(t)), \end{aligned} \quad (7.5)$$

and where (v_+, v_-) denotes now the solution operator for (7.3). Since we have

$$\begin{aligned} \Phi &\in C([0, T) \times (\mathcal{V} \cap (h^{4+\alpha}(\mathbb{S}))^2), (h^{1+\alpha}(\mathbb{S}))^2), \\ \Phi(t, \cdot) &\in C^\omega(\mathcal{V} \cap (h^{4+\alpha}(\mathbb{S}))^2, (h^{1+\alpha}(\mathbb{S}))^2) \quad \text{for all } t \in [0, T), \end{aligned} \quad (7.6)$$

our next goal is to show that

$$-\partial_{(f, h)}\Phi(0, (f_0, h_0)) \in \mathcal{H}((h^{4+\alpha}(\mathbb{S}))^2, (h^{1+\alpha}(\mathbb{S}))^2). \quad (7.7)$$

In the remaining part we let (v_+, v_-) denote the solution to (7.3) determined by the tuple (f_0, h_0, b_0) with $b_0 := b(0)$. As in the first part, the derivative $\partial_{(f, h)}\Phi(0, (f_0, h_0))$ is a matrix operator, three of its components being given by (3.12) (with (f_*, h_*) replaced by (f_0, h_0)), but with

$$(w_+[f], w_-[f]) := (\partial_f v_+(f_0, h_0, b_0)[f], \partial_f v_-(f_0, h_0, b_0)[f])$$

being the solution to the diffraction problem

$$\left\{ \begin{array}{ll} \mathcal{A}(f_0, h_0)w_+[f] = -\partial_f \mathcal{A}(f_0, h_0)[f]v_+ & \text{in } \Omega_+, \\ \mathcal{A}(f_0)w_-[f] = -\partial_f \mathcal{A}(f_0)[f]v_- & \text{in } \Omega_-, \\ \mathcal{B}(f_0, h_0)w_+[f] - \mathcal{B}(f_0)w_-[f] = -\partial_f \mathcal{B}(f_0, h_0)[f]v_+ + \partial_f \mathcal{B}(f_0)[f]v_- & \text{on } \Gamma_0, \\ w_+[f] - w_-[f] = g(\rho_+ - \rho_-)f + \gamma_f \partial_f \kappa(f_0)[f] & \text{on } \Gamma_0, \\ w_+[f] = 0 & \text{on } \Gamma_1, \\ w_-[f] = 0 & \text{on } \Gamma_{-1}, \end{array} \right. \quad (7.8)$$

and with $(W_+[h], W_-[h]) := (\partial_h v_+(f_0, h_0, b_0)[h], \partial_h v_-(f_0, h_0, b_0)[h])$ solving

$$\left\{ \begin{array}{ll} \mathcal{A}(f_0, h_0)W_+[h] = -\partial_h \mathcal{A}(f_0, h_0)[h]v_+ & \text{in } \Omega_+, \\ \mathcal{A}(f_0)W_-[h] = 0 & \text{in } \Omega_-, \\ \mathcal{B}(f_0, h_0)W_+[h] - \mathcal{B}(f_0)W_-[h] = -\partial_h \mathcal{B}(f_0, h_0)[h]v_+ & \text{on } \Gamma_0, \\ W_+[h] - W_-[h] = 0 & \text{on } \Gamma_0, \\ W_+[h] = g\rho_+ h - \gamma_h \partial_h \kappa(h_0)[h] & \text{on } \Gamma_1, \\ W_-[h] = 0 & \text{on } \Gamma_{-1}. \end{array} \right. \quad (7.9)$$

In the following we prove that the diagonal entries of $\partial_{(f,h)}\Phi(0, (f_0, h_0))$ generate analytic semi-groups when seen as unbounded operators in $h^{1+\alpha}(\mathbb{S})$ with dense domain $h^{4+\alpha}(\mathbb{S})$, while one of the off-diagonal entries is in some sense small (see relation (7.10) below). The analysis is not so involved as in the previous sections: on the one hand, the highest derivatives of f and h in (7.8) and (7.9) are hidden in the Fréchet derivative of the curvature operators (the evolution equation (7.4) has a quasilinear structure now), and, on the other hand, we can rely on previous arguments and computations. Therefore, we do not prove all our statements in detail.

The off-diagonal entry. We claim that for each $\varepsilon \in (0, 1)$, there exists $K_0 = K_0(\varepsilon) > 0$ such that

$$\|\partial_h \Phi_1(0, (f_0, h_0))[h]\|_{1+\alpha} \leq \varepsilon \|h\|_{4+\alpha} + K_0 \|h\|_{1+\alpha} \quad \text{for all } h \in h^{2+\alpha}(\mathbb{S}). \quad (7.10)$$

Recalling (3.12) and observing that

$$\partial_h \kappa(h_0)[h] = \frac{h''}{(1 + h_0'^2)^{3/2}} - \frac{3h_0' h_0'' h'}{(1 + h_0'^2)^{5/2}} \quad \text{for all } h \in h^{4+\alpha}(\mathbb{S}),$$

it is useful to split the solution $(W_+[h], W_-[h])$ of (7.9) as

$$(W_+[h], W_-[h]) = (W_+^1, W_-^1) + (W_+^2, W_-^2),$$

where (W_+^1, W_-^1) is the solution to

$$\left\{ \begin{array}{ll} \mathcal{A}(f_0, h_0)W_+^1 = -\partial_h \mathcal{A}(f_0, h_0)[h]v_+ & \text{in } \Omega_+, \\ \mathcal{A}(f_0)W_-^1 = 0 & \text{in } \Omega_-, \\ \mathcal{B}(f_0, h_0)W_+^1 - \mathcal{B}(f_0)W_-^1 = -\partial_h \mathcal{B}(f_0, h_0)[h]v_+ & \text{on } \Gamma_0, \\ W_+^1 - W_-^1 = 0 & \text{on } \Gamma_0, \\ W_+^1 = g\rho_+ h + 3\gamma_h h_0' h_0'' (1 + h_0'^2)^{-5/2} h' & \text{on } \Gamma_1, \\ W_-^1 = 0 & \text{on } \Gamma_{-1}, \end{array} \right. \quad (7.11)$$

while (W_+^2, W_-^2) solves

$$\left\{ \begin{array}{ll} \mathcal{A}(f_0, h_0)W_+^2 = 0 & \text{in } \Omega_+, \\ \mathcal{A}(f_0)W_-^2 = 0 & \text{in } \Omega_-, \\ \mathcal{B}(f_0, h_0)W_+^2 - \mathcal{B}(f_0)W_-^2 = 0 & \text{on } \Gamma_0, \\ W_+^2 - W_-^2 = 0 & \text{on } \Gamma_0, \\ W_+^2 = -\gamma_h (1 + h_0'^2)^{-3/2} h'' & \text{on } \Gamma_1, \\ W_-^2 = 0 & \text{on } \Gamma_{-1}. \end{array} \right. \quad (7.12)$$

In view of (3.12) we get

$$\begin{aligned} \|\partial_h \Phi_1(0, (f_0, h_0))[h]\|_{1+\alpha} &\leq C(\|\text{tr}_0 \nabla W_-^1\|_{1+\alpha} + \|\text{tr}_0 \nabla W_-^2\|_{1+\alpha}) \\ &\leq C(\|W_-^1\|_{2+\alpha}^{\Omega_-} + \|W_-^2\|_{2+\alpha}^{(1/2)\Omega_-}), \end{aligned} \quad (7.13)$$

where we again use the notation $(1/2)\Omega_- = \mathbb{S} \times (-1/2, 0)$. Due to (3.6) we have

$$\|W_-^1\|_{2+\alpha}^{\Omega_-} \leq C\|h\|_{3+\alpha}. \quad (7.14)$$

Finally, we infer from [1, Theorem 9.3] and the arguments in the proof of Proposition 3.3 that

$$\|W_-^2\|_{2+\alpha}^{(1/2)\Omega_-} \leq C(\|W_+^2\|_0^{\Omega_+} + \|W_-^2\|_0^{\Omega_-}) \leq C(\|W_+^2\|_{2+\alpha/2}^{\Omega_+} + \|W_-^2\|_{2+\alpha/2}^{\Omega_-}) \leq C\|h\|_{4+\alpha/2} \quad (7.15)$$

for all $h \in h^{2+\alpha}(\mathbb{S})$. The relations (7.13)-(7.15) and the interpolation property (4.9) lead us the desired estimate (7.10).

The first diagonal entry. We start by identifying the “leading order part” $\partial_f \Phi_1^\pi \in \mathcal{L}(h^{4+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}))$ of $\partial_f \Phi_1(0, (f_0, h_0))$. This is defined as

$$\partial_f \Phi_1^\pi[f] := -\mathcal{B}(f_0)w_-^\pi[f], \quad f \in h^{4+\alpha}(\mathbb{S}),$$

where $(w_+^\pi[f], w_-^\pi[f])$ denotes the solution to the problem

$$\left\{ \begin{array}{ll} \mathcal{A}_0^\pi(f_0, h_0)w_+^\pi[f] = 0 & \text{in } \Omega_+, \\ \mathcal{A}_0^\pi(f_0)w_-^\pi[f] = 0 & \text{in } \Omega_-, \\ \mathcal{B}(f_0, h_0)w_+^\pi[f] - \mathcal{B}(f_0)w_-^\pi[f] = 0 & \text{on } \Gamma_0, \\ w_+^\pi[f] - w_-^\pi[f] = \gamma_f(1 + f_0'^2)^{-3/2}f'' & \text{on } \Gamma_0, \\ w_+^\pi[f] = 0 & \text{on } \Gamma_1, \\ w_-^\pi[f] = 0 & \text{on } \Gamma_{-1}. \end{array} \right.$$

Here, $\mathcal{A}_0^\pi(f_0)$ and $\mathcal{A}_0^\pi(f_0, h_0)$ are the operators defined by (5.3) (with (f_*, h_*) replaced by (f_0, h_0)). The same arguments as in the proof of Lemma 5.1 show that, given $\varepsilon \in (0, 1)$, there exists $K_1 = K_1(\varepsilon) > 0$ such that

$$\|\partial_f \Phi_1(0, (f_0, h_0))[f] - \partial_f \Phi_1^\pi[f]\|_{1+\alpha} \leq \varepsilon \|f\|_{4+\alpha} + K_1 \|f\|_{1+\alpha} \quad \text{for all } f \in h^{4+\alpha}(\mathbb{S}). \quad (7.16)$$

Theorem 7.2. *Let $\mu > 0$ and $\alpha' \in (0, \alpha)$ be given. Then there exist an integer $p \geq 3$, a p -partition of unity $\{\Pi_j^p\}_{1 \leq j \leq 2^{p+1}}$, and a constant $K_2 = K_2(p)$, and for each $1 \leq j \leq 2^{p+1}$ there are bounded operators $\mathbb{A}_j \in \mathcal{L}(h^{4+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}))$ such that*

$$\|\Pi_j^p \partial_f \Phi_1^\pi[f] - \mathbb{A}_j[\Pi_j^p f]\|_{1+\alpha} \leq \mu \|\Pi_j^p f\|_{4+\alpha} + K_2 \|f\|_{4+\alpha'}$$

for all $f \in h^{4+\alpha}(\mathbb{S})$. The operators \mathbb{A}_j are defined by the formula

$$\mathbb{A}_j[f] := -\frac{k}{\mu_-} \left(\frac{1 + f_0'^2}{f_0 - d} \Big|_{x_j^p} \text{tr}_0 \partial_y w_-^{\pi, j}[f] - f_0'(x_j^p) \text{tr}_0 \partial_x w_-^{\pi, j}[f] \right),$$

where $(w_+^{\pi, j}[f], w_-^{\pi, j}[f])$ denotes the solution to the problem

$$\left\{ \begin{array}{ll} \mathcal{A}_{0,j}^\pi(f_0, h_0)w_+^{\pi, j}[f] = 0 & \text{in } \Omega_+, \\ \mathcal{A}_{0,j}^\pi(f_0)w_-^{\pi, j}[f] = 0 & \text{in } \Omega_-, \\ \mathcal{B}_j(f_0, h_0)w_+^{\pi, j}[f] - \mathcal{B}_j(f_0)w_-^{\pi, j}[f] = 0 & \text{on } \Gamma_0, \\ w_+^{\pi, j}[f] - w_-^{\pi, j}[f] = V_f f'' & \text{on } \Gamma_0, \\ w_+^{\pi, j}[f] = 0 & \text{on } \Gamma_1, \\ w_-^{\pi, j}[f] = 0 & \text{on } \Gamma_{-1}. \end{array} \right.$$

The operators $\mathcal{A}_{0,j}^\pi(f_0, h_0)$, $\mathcal{A}_{0,j}^\pi(f_0)$, $\mathcal{B}_j(f_0, h_0)$, and $\mathcal{B}_j(f_0)$ are the same as in Theorem 5.5 (with (f_*, h_*) replaced by (f_0, h_0)) and $V_f := \gamma_f(1 + f_0'^2(x_j^p))^{-3/2}$.

Proof. The proof is similar to that of Theorem 5.5 and is therefore omitted. \square

From the computations in Section 5 one infers that, given a p -partition of unity $\{\Pi_j^p\}_{1 \leq j \leq 2^{p+1}}$ and $1 \leq j \leq 2^{p+1}$, the operator \mathbb{A}_j is a Fourier multiplier of the form

$$\mathbb{A}_j \left[\sum_{m \in \mathbb{Z}} f_m e^{imx} \right] = \sum_{m \in \mathbb{Z}} \tilde{\lambda}_m f_m e^{imx},$$

the (real) symbol $(\tilde{\lambda}_m)_m$ being given by

$$\tilde{\lambda}_m = -V_f \left(\frac{\tanh(D_+ m)}{\beta_2^+ D_+ m} + \frac{\tanh(D_- m)}{\beta_2^- D_- m} \right)^{-1} m^2,$$

cf. (5.23) and (5.24). Here, β_2^\pm and D_\pm are the positive constants defined in Section 5 (with (f_*, h_*) replaced by (f_0, h_0)).

Lemma 7.3. *Given $(f_0, h_0) \in \mathcal{V} \cap (h^{4+\alpha}(\mathbb{S}))^2$ and $b_0 \in h^{2+\alpha}(\mathbb{S})$, there exist constants $\kappa_1 \geq 1$ and $\omega_1 > 0$ depending only on (f_0, h_0, b_0) such that for any p -partition of unity $\{\Pi_j^p\}_{1 \leq j \leq 2^{p+1}}$ with $p \geq 3$, the operators \mathbb{A}_j , $1 \leq j \leq 2^{p+1}$, satisfy*

$$\begin{aligned} \lambda - \mathbb{A}_j &\in \text{Isom}(h^{4+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})), \\ \kappa_1 \|(\lambda - \mathbb{A}_j)[f]\|_{1+\alpha} &\geq |\lambda| \cdot \|f\|_{1+\alpha} + \|f\|_{4+\alpha} \end{aligned}$$

for all $f \in h^{4+\alpha}(\mathbb{S})$ and $\lambda \in \mathbb{C}$ with $\text{Re } \lambda \geq \omega_1$.

Proof. This follows analogously as in the proof of Lemma 5.6. \square

We are now in the position to prove the generator property for $\partial_f \Phi_1(0, (f_0, h_0))$.

Theorem 7.4. *If $(f_0, h_0) \in \mathcal{V} \cap (h^{4+\alpha}(\mathbb{S}))^2$ and $b_0 \in h^{2+\alpha}(\mathbb{S})$, then*

$$-\partial_f \Phi_1(0, (f_0, h_0)) \in \mathcal{H}(h^{4+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})).$$

Proof. Because of (7.16) and [2, Theorem I.1.3.1 (ii)], we only need to prove that $-\partial_f \Phi_1^\pi$ is an analytic generator. The arguments in the proof of Theorem 5.7 together with Remark 5.4, Theorem 7.2, and Lemma 7.3 show that there exist constants $\tilde{\kappa}_1 > 0$ and $\tilde{\omega}_1 > 1$ such that

$$\tilde{\kappa}_1 \|(\lambda - \partial_f \Phi_1^\pi)[f]\|_{1+\alpha} \geq |\lambda| \cdot \|f\|_{1+\alpha} + \|f\|_{4+\alpha} \quad (7.17)$$

for all $\lambda \in \mathbb{C}$ with $\text{Re } \lambda \geq \tilde{\omega}_1$ and $f \in h^{4+\alpha}(\mathbb{S})$. As $(\lambda - \partial_f \Phi_1^\pi)$ is one-to-one for $\text{Re } \lambda \geq \tilde{\omega}_1$ by (7.17), we are left to show that $(\lambda - \partial_f \Phi_1^\pi)$ is a Fredholm operator of index zero for $\lambda \geq \tilde{\omega}_1$, see [2, Remark I.1.2.1].

We can decompose $\partial_f \Phi_1^\pi$ as $\partial_f \Phi_1^\pi = \Psi_1 \circ \Psi_2$, where $\Psi_2 : h^{4+\alpha}(\mathbb{S}) \rightarrow h^{2+\alpha}(\mathbb{S})$ is defined by

$$\Psi_2 f := -\frac{\gamma f}{(1 + f_0'^2)^{3/2}} f'',$$

and $\Psi_1 : h^{2+\alpha}(\mathbb{S}) \rightarrow h^{1+\alpha}(\mathbb{S})$ is the operator obtained from $\partial_f \Phi_{1,0}^\pi[f]$ when choosing $\Delta_\rho = 1$, cf. (5.7) and (5.8). A simple consequence of Proposition 5.8 is that Ψ_1 is a Fredholm operator of index zero, the same Fredholm property being valid also for Ψ_2 . By a classical result [34, Theorem 13.1] we then know that also $\partial_f \Phi_1^\pi = \Psi_1 \circ \Psi_2$ is also a Fredholm operator of index zero. This property allows us to conclude that $(\lambda - \partial_f \Phi_1^\pi)$ is bijective for all $\lambda \geq \tilde{\omega}_1$, and to finish the proof. \square

Remark 7.5. *Let $(f_*, h_*) \in \mathcal{V}$ be given and $a \in h^{2+\alpha}(\mathbb{S})$ be a negative function. Moreover, let \mathbb{A} be the operator defined by (5.7). Then,*

$$-\mathbb{A} \circ \partial_x^2 \in \mathcal{H}(h^{4+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})).$$

The second diagonal entry. We define the operator $\partial_h \Phi_2^\pi \in \mathcal{L}(h^{4+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}))$ by the formula

$$\partial_h \Phi_2^\pi[h] := -\mathcal{B}_1(f_0, h_0) W_+^\pi[h], \quad h \in h^{4+\alpha}(\mathbb{S}),$$

where $W_+^\pi[h]$ denotes the solution to the problem

$$\begin{cases} \mathcal{A}_1^\pi(f_0, h_0)W_+^\pi[h] = 0 & \text{in } \Omega_+, \\ W_+^\pi[h] = 0 & \text{on } \Gamma_0, \\ W_+^\pi[h] = -\gamma_h(1 + h_0'^2)^{-3/2}h'' & \text{on } \Gamma_1, \end{cases}$$

and with $\mathcal{A}_1^\pi(f_0, h_0)$ defined by (6.3) (with (f_*, h_*) replaced by (f_0, h_0)). The operator $\partial_h \Phi_2^\pi$ is the “leading order” part of the Fréchet derivative $\partial_h \Phi_2(0, (f_0, h_0))$ in the sense that, given $\varepsilon \in (0, 1)$, there exists $K_3 = K_3(\varepsilon) > 0$ such that

$$\|\partial_h \Phi_2(0, (f_0, h_0))[h] - \partial_h \Phi_2^\pi[h]\|_{1+\alpha} \leq \varepsilon \|h\|_{4+\alpha} + K_3 \|h\|_{1+\alpha} \quad \text{for all } h \in h^{4+\alpha}(\mathbb{S}). \quad (7.18)$$

The estimate (7.18) is obtained by arguing as in Lemma 6.1.

Theorem 7.6. *Let $\mu > 0$ and $\alpha' \in (0, \alpha)$ be given. Then there exist an integer $p \geq 3$, a p -partition of unity $\{\Pi_j^p\}_{1 \leq j \leq 2^{p+1}}$, and a constant $K_4 = K_4(p)$, and for each $1 \leq j \leq 2^{p+1}$ there are bounded operators $\mathbb{B}_j \in \mathcal{L}(h^{4+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}))$ such that*

$$\|\Pi_j^p \partial_h \Phi_2^\pi[h] - \mathbb{B}_j[\Pi_j^p h]\|_{1+\alpha} \leq \mu \|\Pi_j^p h\|_{4+\alpha} + K_4 \|h\|_{4+\alpha'}$$

for all $h \in h^{4+\alpha}(\mathbb{S})$. The operators \mathbb{B}_j are defined by the formula

$$\mathbb{B}_j[h] := -\frac{k}{\mu_+} \left(\frac{1 + h_0'^2}{h_0 - f_0} \Big|_{x_j^p} \text{tr}_1 \partial_y W_+^{\pi,j}[h] - h_0'(x_j^p) \text{tr}_1 \partial_x W_+^{\pi,j}[h] \right),$$

where $W_+^{\pi,j}[h]$ denotes the solution to the problem

$$\begin{cases} \mathcal{A}_{1,j}^\pi(f_0, h_0)W_+^{\pi,j}[h] = 0 & \text{in } \Omega_+, \\ W_+^{\pi,j}[h] = 0 & \text{on } \Gamma_0, \\ W_+^{\pi,j}[h] = -V_h h'' & \text{on } \Gamma_1, \end{cases}$$

with $\mathcal{A}_{1,j}^\pi(f_0, h_0)$ as in Theorem 6.3 (with (f_*, h_*) replaced by (f_0, h_0)) and $V_h := \gamma_h(1 + h_0'^2(x_j^p))^{-3/2}$.

Proof. The proof is similar to that of Theorem 6.3 and is therefore omitted. \square

The computations in Section 6 show that, given a p -partition of unity $\{\Pi_j^p\}_{1 \leq j \leq 2^{p+1}}$ and $1 \leq j \leq 2^{p+1}$, the operator \mathbb{B}_j is a Fourier multiplier of the form

$$\mathbb{B}_j \left[\sum_{m \in \mathbb{Z}} h_m e^{imx} \right] = \sum_{m \in \mathbb{Z}} \tilde{\varphi}_m h_m e^{imx},$$

the (real) symbol $(\tilde{\varphi}_m)_m$ being given by

$$\tilde{\varphi}_m = -\frac{kV_h}{\mu_+} \frac{m^3}{\tanh(Dm)},$$

cf. (6.18) and (6.19), where D is the positive constant defined in Section 6 (with (f_*, h_*) replaced by (f_0, h_0)).

Lemma 7.7. *Given $(f_0, h_0) \in \mathcal{V} \cap (h^{4+\alpha}(\mathbb{S}))^2$ and $b_0 \in h^{2+\alpha}(\mathbb{S})$, there exist constants $\kappa_2 \geq 1$ and $\omega_2 > 0$ depending only on (f_0, h_0, b_0) such that for any p -partition of unity $\{\Pi_j^p\}_{1 \leq j \leq 2^{p+1}}$ with $p \geq 3$, the operators \mathbb{B}_j , $1 \leq j \leq 2^{p+1}$, satisfy*

$$\lambda - \mathbb{B}_j \in \text{Isom}(h^{4+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})),$$

$$\kappa_2 \|(\lambda - \mathbb{B}_j)[h]\|_{1+\alpha} \geq |\lambda| \cdot \|h\|_{1+\alpha} + \|h\|_{4+\alpha}$$

for all $h \in h^{4+\alpha}(\mathbb{S})$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geq \omega_2$.

Proof. This follows analogously as in the proof of Lemma 6.4. \square

We now establish the generator property for $\partial_h \Phi_2(0, (f_0, h_0))$.

Theorem 7.8. *If $(f_0, h_0) \in \mathcal{V} \cap (h^{4+\alpha}(\mathbb{S}))^2$ and $b_0 \in h^{2+\alpha}(\mathbb{S})$, then*

$$-\partial_h \Phi_2(0, (f_0, h_0)) \in \mathcal{H}(h^{4+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})).$$

Proof. The proof follows by combining the arguments used to establish Theorems 6.5 and 7.4. \square

We conclude this section with the proof of Theorem 7.1.

Proof of Theorem 7.1. In view of Theorems 7.4, 7.8, relation (7.10) and [2, Theorem I.1.6.1 and Remark I.1.6.2]), we get that

$$-\partial_{(f,h)} \Phi(0, (f_0, h_0)) \in \mathcal{H}((h^{4+\alpha}(\mathbb{S}))^2, (h^{1+\alpha}(\mathbb{S}))^2)$$

for all $(f_0, h_0) \in \mathcal{V} \cap (h^{4+\alpha}(\mathbb{S}))^2$ and $b_0 = b(0) \in h^{2+\alpha}(\mathbb{S})$. The remaining part of the proof is identical to that of Theorem 2.1. \square

APPENDIX A.

We collect here some explicit formulae for operators used in the previous sections. Given $(f, h) \in \mathcal{V}$, it follows readily from the definitions in Section 3 of the operators $\mathcal{A}(f)$, $\mathcal{A}(f, h)$, $\mathcal{B}(f)$, and $\mathcal{B}(f, h)$ that

$$\begin{aligned} \mathcal{A}(f) &= \partial_{xx} - 2 \frac{(1+y)f'}{f-d} \partial_{xy} + \frac{(1+y)^2 f'^2 + 1}{(f-d)^2} \partial_{yy} - (1+y) \frac{(f-d)f'' - 2f'^2}{(f-d)^2} \partial_y, \\ \mathcal{A}(f, h) &= \partial_{xx} - 2 \frac{yh' + (1-y)f'}{h-f} \partial_{xy} + \frac{(yh' + (1-y)f')^2 + 1}{(h-f)^2} \partial_{yy} - \left[\frac{yh'' + (1-y)f''}{h-f} \right. \\ &\quad \left. - 2 \frac{(h' - f')(yh' + (1-y)f')}{(h-f)^2} \right] \partial_y, \\ \mathcal{B}(f) &= \frac{k}{\mu_-} \left(\frac{1+f'^2}{f-d} \operatorname{tr}_0 \partial_y - f' \operatorname{tr}_0 \partial_x \right), \quad \mathcal{B}(f, h) = \frac{k}{\mu_+} \left(\frac{1+f'^2}{h-f} \operatorname{tr}_0 \partial_y - f' \operatorname{tr}_0 \partial_x \right), \\ \mathcal{B}_1(f, h) &= \frac{k}{\mu_+} \left(\frac{1+h'^2}{h-f} \operatorname{tr}_1 \partial_y - h' \operatorname{tr}_1 \partial_x \right), \end{aligned}$$

and therefore their (partial) derivatives at a fixed $(f_*, h_*) \in \mathcal{V}$ are given by

$$\begin{aligned}\partial_f \mathcal{A}(f_*)[f] &= 2 \left[\frac{(1+y)f'_* f}{(f_* - d)^2} - \frac{(1+y)f'}{f_* - d} \right] \partial_{xy} + 2 \left[\frac{(1+y)^2 f'_* f'}{(f_* - d)^2} - \frac{(1+y)^2 f'^2_* + 1}{(f_* - d)^3} f \right] \partial_{yy} \\ &\quad - (1+y) \left[\frac{(f_* - d)f'' + f'_* f' - 4f'_* f'}{(f_* - d)^2} - 2 \frac{(f_* - d)f''_* - 2f'^2_*}{(f_* - d)^3} f \right] \partial_y, \\ \partial_f \mathcal{A}(f_*, h_*)[f] &= -2 \left[\frac{(1-y)f'}{h_* - f_*} + \frac{yh'_* + (1-y)f'_* f}{(h_* - f_*)^2} \right] \partial_{xy} + \left[2 \frac{(yh'_* + (1-y)f'_*)^2 + 1}{(h_* - f_*)^3} f \right. \\ &\quad \left. + \frac{(1-y)(yh'_* + (1-y)f'_*) f'}{(h_* - f_*)^2} \right] \partial_{yy} - \left[\frac{(1-y)f''}{h_* - f_*} + \frac{yh''_* + (1-y)f''_*}{(h_* - f_*)^2} f \right. \\ &\quad \left. - 2 \frac{(1-2y)(h'_* - f'_*) - f'_* f'}{(h_* - f_*)^2} f' - 4 \frac{(h'_* - f'_*)(yh'_* + (1-y)f'_*)}{(h_* - f_*)^3} f \right] \partial_y, \\ \partial_h \mathcal{A}(f_*, h_*)[h] &= 2 \left[\frac{yh'_* + (1-y)f'_*}{(h_* - f_*)^2} h - \frac{yh'}{h_* - f_*} \right] \partial_{xy} + 2 \left[\frac{y(yh'_* + (1-y)f'_*)}{(h_* - f_*)^2} h' \right. \\ &\quad \left. - \frac{(yh'_* + (1-y)f'_*)^2 + 1}{(h_* - f_*)^3} h \right] \partial_{yy} - \left[\frac{yh''}{h_* - f_*} - \frac{yh''_* + (1-y)f''_*}{(h_* - f_*)^2} h \right. \\ &\quad \left. - 2 \frac{2yh'_* + (1-2y)f'_*}{(h_* - f_*)^2} h' - 4 \frac{(h'_* - f'_*)(yh'_* + (1-y)f'_*)}{(h_* - f_*)^3} h \right] \partial_y,\end{aligned}$$

respectively by

$$\begin{aligned}\partial_f \mathcal{B}(f_*)[f] &= \frac{k}{\mu_-} \left[\left(\frac{2f'_* f'}{f_* - d} - \frac{1 + f'^2_*}{(f_* - d)^2} f \right) \text{tr}_0 \partial_y - f' \text{tr}_0 \partial_x \right], \\ \partial_f \mathcal{B}(f_*, h_*)[f] &= \frac{k}{\mu_+} \left[\left(\frac{2f'_* f'}{h_* - f_*} + \frac{1 + f'^2_*}{(h_* - f_*)^2} f \right) \text{tr}_0 \partial_y - f' \text{tr}_0 \partial_x \right], \\ \partial_h \mathcal{B}(f_*, h_*)[h] &= -\frac{k}{\mu_+} \frac{1 + f'^2_*}{(h_* - f_*)^2} h \text{tr}_0 \partial_y, \\ \partial_h \mathcal{B}_1(f_*, h_*)[h] &= \frac{k}{\mu_+} \left[\left(\frac{2h'_*}{h_* - f_*} h' - \frac{1 + h'^2_*}{(h_* - f_*)^2} h \right) \text{tr}_1 \partial_y - h' \text{tr}_1 \partial_x \right]\end{aligned}$$

for $(f, h) \in (h^{2+\alpha}(\mathbb{S}))^2$.

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